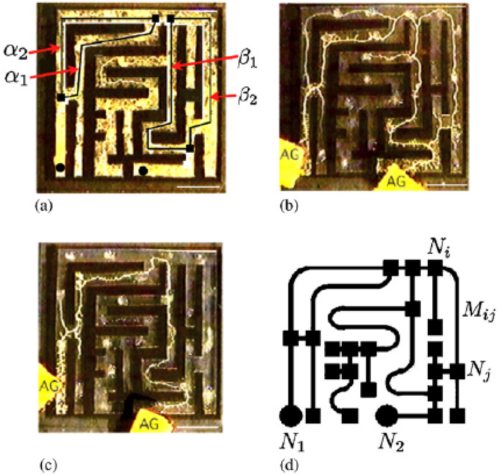
Physarum Computations

Luca Becchetti, Ruben Becker, Vincenzo Bonifaci,
Michael Dirnberger, Andreas Karrenbauer, Pavel Kolev,
Kurt Mehlhorn, Girish Varma

SODA 2012, ICALP 2013, J. Theoretical Biology 2012,
Journal of Physics D: Applied Physics 2017,Arxive 2017

max planck institut
informatik

Physarum solves shortest path problems



Physarum, a slime mold,

single cell, several nuclei

builds evolving networks

Nakagaki, Yamada, Tóth, Nature 2000

show video

For achievements that first make people LAUGH
then make them THINK

COGNITIVE SCIENCE PRIZE: Toshiyuki Nakagaki, Ryo Kobayashi, Atsushi Tero, Ágotá Tóth
for discovering that slime molds can solve puzzles.

REFERENCE: "Intelligence: Maze-Solving by an Amoeboid Organism," Toshiyuki Nakagaki, Hiroyasu Yamada, and Ágota Tóth, Nature, vol. 407, September 2000, p. 470.

Outline of Talk

- The maze experiment (Nakagaki, Yamada, Tóth). ✓
- A mathematical model for the dynamics of Physarum (Tero et al.).
- The result: convergence against the shortest path.
- Approach:
  - Analytical investigation of simple systems.
  - A simulator.
  - Formulizing conjectures and killing them.
  - Proving the surviving conjecture.
- Beyond shortest paths.
  - Transportation problems.
  - Linear programming
  - Network formation.

Mathematical Model (Tero et al.)

- Physarum is a network of tubes (pipes);
- Flow (of liquids and nutrients) through a tube is determined by concentration differences at endpoints of a tube, length of tube, and diameter of tube;
- Tubes adapt to the flow through them: if flow through a tube is high (low) relative to diameter of the tube, the tube grows (shrinks) in diameter.
- Mathematics is the same as for flows in an electrical network with time-dependent resistors.
- Tero et al., J. of Theoretical Biology, 553 – 564, 2007

Mathematical Model (Tero et al.)

- $G = (V, E)$ undirected graph
- Each edge e has a positive length c_e (fixed) and a positive diameter $x_e(t)$ (dynamic).
- Send one unit of current (flow) from s_0 to s_1 in an electrical network where resistance of e equals

$$r_e(t) = c_e / x_e(t).$$

- $q_e(t)$ is resulting flow across e at time t.
- Dynamics:

$$\dot{x}_e(t) = \frac{dx_e(t)}{dt} = |q_e(t)| - x_e(t).$$

We will write x_e and q_e instead of $x_e(t)$ and $q_e(t)$ from now on.

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The Questions

Does the system convergence for all (!!!) initial conditions?

If so, what does it converge to? Fixpoints?

How fast does it converge?

Beyond shortest paths?

Inspiration for distributed algorithms?

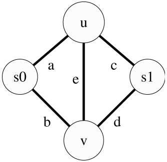
Convergence against Shortest Path

Theorem (Convergence (SODA 12, J. Theoretical Biology))

Dynamics converge against shortest path, i.e.,

- *potential difference between source and sink converges to length of shortest source-sink path,*
- $x_e \to 1$ *for edges on shortest source-sink path,*
- $x_e \to 0$ *for edges not on shortest source sink path*

this assumes that shortest path is unique; otherwise ...



Miyaji/Onishi previously proved convergence for parallel links and Wheatstone graph.

Our Approach

- Analytical investigation of simple systems, in particular, parallel links, and
- experimental investigation (computer simulation) of larger systems,
  - to form intuition about the dynamics,
  - to kill conjectures,
  - to support conjectures.
- Proof attempts for conjectures surviving experiments

Computer Simulation (Discrete Time)

- Electrical flows are driven by electrical potentials; let p_u be the potential at node u at time t. ($p_{s_1} = 0$ always)
- $q_e = x_e(p_u - p_v)/c_e$ is flow on edge $\{ u, v \}$ from u to v.
- Flow conservation gives n equations.

$$\text{for all vertices } u: \quad \sum_{v;\ e=\{ u,v \} \in E} x_e(p_u - p_v)/c_e = b_u.$$

- $b_{s_0} = 1 = -b_{s_1}$ and $b_u = 0$, otherwise.
- The equations above define the p_v's and the q_e's uniquely and can be computed by solving a linear system.
- Discrete Dynamics: $x_e(t+1) = x_e(t) + h \cdot (|q_e(t)| - x_e(t))$.

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- Remark: linear system best solved by iterative method; simulation requires arbitrary precision arithmetic.

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- We simulated 1000 systems with up to 10000 nodes. Always observed convergence to shortest path. Speed of convergence is determined by ratio of length of second shortest path to length of shortest path.

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- from now on: $\Delta_e = p_u - p_v$ for $e = uv$; potential drop on e.

The General Case

Fixpoints: It is easy to verify (quarter page) that the fixpoints are exactly the source-sink paths. This assumes that all paths have different length. Thus, if the system converges, it converges against some source-sink path.

Convergence:

- In order to prove convergence, one needs to find a Lyapunov function, i.e., a function L mapping x to real numbers such that
  - $L(x) \geq 0$ for all x,
  - $\frac{d}{dt}L(x) \leq 0$, and
  - $\dot{L} = 0$ if and only if $\dot{x} = 0$.
- In order to prove convergence against the shortest path, one needs some additional arguments.

Lyapunov Functions?

First idea: the energy of the flow $\sum_e q_e \Delta_e$ decreases over. time

Not true, even for parallel links.

Theorem

For the case of parallel links:

$$\sum_{i \geq 2} c_i \ln x_i - c_1 \ln x_1 , \sum_i q_i c_i , \quad \frac{\sum_i x_i c_i}{\sum_i x_i} , \quad \textit{and} \, (p_s - p_t) \sum_i x_i c_i$$

decrease over time

computer experiment: the obvious generalizations to general graphs (replace i by e) do not work.

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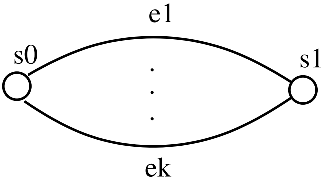
A not so Obvious Generalization



$$\frac{\sum_i x_i c_i}{\sum_i x_i} \quad \Rightarrow \quad \frac{\sum_e x_e c_e}{\text{minimum total diameter of a } s_0\text{-}s_1 \text{ cut}}$$

LEDA came handy.

Lyapunov Functions?

Computer experiment:

$$V := \frac{\sum_e x_e c_e}{\text{minimum total diameter of a } s_0\text{-}s_1 \text{ cut}} \quad \text{decreases}$$

Theorem (Lyapunov Function)

$$V + \left(\sum_{e \in \delta(\{ s_0 \})} x_e - 1\right)^2 \quad \textit{decreases.}$$

Derivative of V (essentially) satisfies

$$\dot{V} \le -c \cdot \sum_e (x_e - |q_e|)^2.$$

Proof uses min-cut-max-flow and ...

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Convergence against Shortest Path

Corollary (Convergence)

Dynamics converge against shortest path, i.e.,

- *potential difference between source and sink converges to length of shortest source-sink path,*
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this assumes that shortest path is unique; otherwise, ...

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Discretization and Speed of Convergence

$$x_e(t+1) = x_e(t) + h(|q_e(t)| - x_e(t))$$

Theorem (Epsilon-Approximation of Shortest Path)

Let *opt* be the length of the shortest source-sink path.

Let $\varepsilon > 0$ be arbitrary. Set $h = \varepsilon/(2mL)$, where L is largest edge length and m is the number of edges.

After $\widetilde{O}(nmL^2/\varepsilon^3)$ iterations, solution is $(1+\varepsilon)$ optimal, i.e., $V = \sum_e c_e x_e$ is at most $(1+\varepsilon)opt$.

Arithmetic with $O(\log(nL/\varepsilon))$ bits suffices.

A Generalization (Arxive 2017)

Recall: The dynamics $\dot{x} = |q| - x$

- q is the electrical flow with respect to resistances $r_e = c_e/x_e$. One unit of flow from source to sink.
- x converges to undirected shortest source-sink path.

- alternative formulation: A = node-edge incidence matrix of an arbitrary orientation of G, b demand vector (-1 is row s_0 and $+1$ in row s_1). Then (Thomson's principle)

  $$q \text{ is the solution of } Af = b \text{ minimizing the energy } f^T Rf,$$

  where $R = diag(c_e/x_e)$.
- shortest undirected path = minimize $c^T|x|$ subject to $Ax = b$.
- What happens if A is a general matrix?

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Nonnegative Linear Programs

The dynamics $\dot{x} = |q| - x$

- $A \in \mathbb{R}^{n \times m}$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}^m_{\geq 0}$. Every vector f in the kernel of A, i.e., $Af = 0$, has positive cost $c^T|f| > 0$.
- q is the solution of $Af = b$ minimizing $f^T Rf$, where $R = diag(c_e/x_e)$.
- Theorem: $x(t)$ converges to an optimal solution of

$$\text{minimize } c^T|x| \text{ subject to } Ax = b.$$

This assumes that the optimal solution is unique. Otherwise...

Discretization was shown to converge by Straszak and Vishnoi.

Related Work: Directed Physarum

$$\dot{D}_e(t) = q_e(t) - x_e(t)$$

No biological significance is claimed.

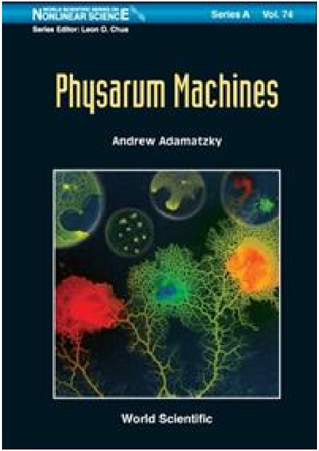
Results

Ito/Johansson/Nakagaki/Tero (2011) prove convergence to shortest directed source-sink path.

Johannson/Zou (2012) and D. Straszak/N. Vishnoi (2016) prove that directed dynamics solves any linear program with monotone objective function (all coefficients of c are positive)

$$\max c^T x \quad \text{subject to} \quad Ax = b \text{ and } x \geq 0.$$

Adamatzky's Book



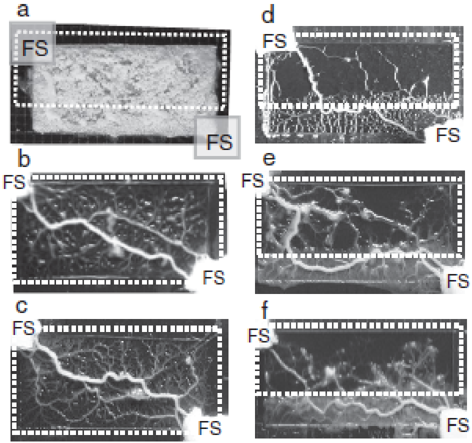
many examples of Physarum computations

- shortest paths
- network design
- Delaunay diagrams
- puzzles

also Youtube-videos: search for Physarum

Open Problems

Nonuniform Physarum



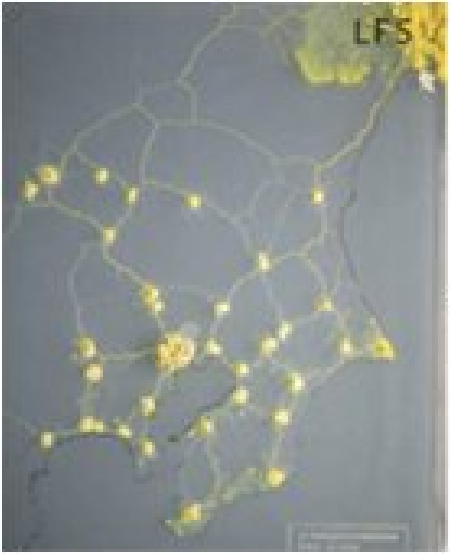
$$\dot{D}_e(t) = a_e(|q_e(t)| - x_e(t))$$

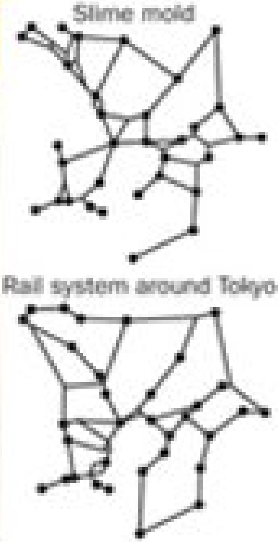
a_e reactivity of e

We have a heuristic argument for the details of the convergence process. Have verified them in computer simulations.

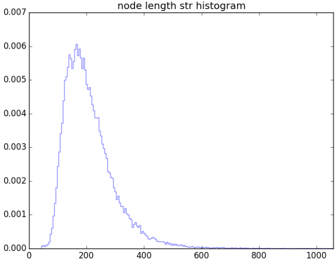
No convergence proof

Network Design: Science 2010

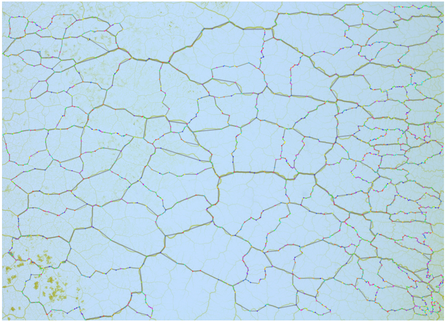




Observables Demo



Histogram for edge lengths. Abscissa shows values in pixel.



Have verified experimentally that cut capacity orthogonal to growing direction is constant.

Dirnberger/Mehlhorn/Mehlhorn, J. Phys. D, 2017, Dirnberger/Mehlhorn, J. Phys. D, 2017.

My Current Projects

Understand the principles of network formation. What does the network optimize?

Nonuniform Versions of Physarum.

Can I use Physarum as an inspiration for approximation algorithms?