Power of Algorithms
in Discrete Optimisation

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What captures efficient computation?
What captures efficient computation?

What are the limits of efficient algorithms?
convex relaxations
(s,t)-Min-Cut
(s,t)-Min-Cut

\[
\min_{x_1, \ldots, x_n \in \{0,1\}, s=0, t=1} \sum_{\{i,j\} \in E(G)} \phi(x_i, x_j)
\]
(s,t)-Min-Cut

\[
\min_{x_1, \ldots, x_n \in \{0,1\}} \sum_{s=0, t=1}^{\{i,j\} \in E(G)} \phi(x_i, x_j)
\]
(s,t)-Min-Cut

$$\begin{array}{c|c|c|c}
 x & y & \phi(x, y) \\
 \hline
 0 & 0 & 0 \\
 0 & 1 & 1 \\
 1 & 0 & 1 \\
 1 & 1 & 0 \\
\end{array}$$

$$\min_{x_1, \ldots, x_n \in \{0,1\}} \sum_{s=0, t=1}^{i,j \in E(G)} \phi(x_i, x_j)$$

$$\phi(0, 0) + \phi(1, 1) \leq \phi(0, 1) + \phi(1, 0)$$
(s,t)-Min-Cut

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\begin{array}{c|c|c}
  x & y & \phi(x,y) \\
  \hline
  0 & 0 & 0 \\
  0 & 1 & 1 \\
  1 & 0 & 1 \\
  1 & 1 & 0 \\
\end{array}
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\min_{x_1,\ldots,x_n \in \{0,1\}} \sum_{s=0, t=1, \{i,j\} \in E(G)} \phi(x_i, x_j)
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\phi(0, 0) + \phi(1, 1) \leq \phi(0, 1) + \phi(1, 0)
\]

submodularity
(s,t)-Min-Cut

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\begin{array}{c|c|c|c}
  x & y & \phi(x, y) \\
  \hline
  0 & 0 & 0 \\
  0 & 1 & 1 \\
  1 & 0 & 1 \\
  1 & 1 & 0 \\
\end{array}
\]

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\min_{x_1, \ldots, x_n \in \{0,1\}} \sum_{s=0, t=1, \{i,j\} \in E(G)} \phi(x_i, x_j)
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submodularity
(s,t)-Min-Cut

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\min_{x_1, \ldots, x_n \in \{0,1\}} \sum_{s=0, t=1}^{\{i,j\} \in E(G)} \phi(x_i, x_j)
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\phi(0,0) + \phi(1,1) \leq \phi(0,1) + \phi(1,0)
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**submodularity**
(s,t)-Min-Cut

**Thm:** CSPs with $\mathbb{Q}$-valued fns in PTIME if ..., NP-complete otherwise

[Thapper, Ž. JACM’16]
(s,t)-Min-Cut

Thm: CSPs with \( \mathbb{Q} \)-valued fns in PTIME if ..., NP-complete otherwise

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\min_{x_1, \ldots, x_n \in \{0,1\}} \sum_{s=0, t=1, \{i,j\} \in E(G)} \phi(x_i, x_j)
\]

\[
\phi(0, 0) + \phi(1, 1) \leq \phi(0, 1) + \phi(1, 0)
\]

submodularity

dichotomy!

[Thapper, Ž. JACM’16]
(s,t)-Min-Cut

\[
\min_{x_1, \ldots, x_n \in \{0,1\}} \sum_{s=0, t=1}^{E(G)} \phi(x_i, x_j)
\]

\[
\phi(0, 0) + \phi(1, 1) \leq \phi(0, 1) + \phi(1, 0)
\]

Thm: CSPs with \(\mathbb{Q}\)-valued fns in PTIME if ..., NP-complete otherwise

[Thapper, Ž. JACM’16]
Thm: CSPs with $\mathbb{Q}$-valued fns in PTIME if ..., NP-complete otherwise (all tractable cases via Basic LP)

\begin{align*}
\min_{x_1, \ldots, x_n \in \{0,1\}} & \sum_{s=0, t=1} \sum_{i,j \in E(G)} \phi(x_i, x_j) \\
\text{subject to} & \\
\phi(0,0) + \phi(1,1) \leq \phi(0,1) + \phi(1,0)
\end{align*}

Thapper, Ž. JACM'16

(s,t)-Min-Cut

\begin{tabular}{|c|c|c|}
\hline
$x$ & $y$ & $\phi(x,y)$ \\
\hline
0 & 0 & 0 \\
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0 \\
\hline
\end{tabular}

submodularity

dichotomy!
Convex Relaxations
Convex Relaxations

$$\min_{x \in \Omega} f(x)$$
Convex Relaxations

$$\min_{x \in \Omega} f(x) \quad \rightarrow \quad \min_{x \in \Omega'} f'(x)$$
Convex Relaxations

\[
\min_{x \in \Omega} f(x) \quad \Rightarrow \quad \min_{x \in \Omega'} f'(x)
\]
Convex Relaxations

\[
\min_{x \in \Omega} f(x)
\]

\[
\min_{x \in \Omega'} f'(x)
\]

\[\text{opt} \quad \Rightarrow \quad \text{opt'}\]
Convex Relaxations

Linear (LP) and Semidefinite (SDP) Programming
Convex Relaxations

\[ \min_{x \in \Omega} f(x) \]

LP/SDP

\[ \min_{x \in \Omega'} f'(x) \]

\[ \text{opt} \]

\[ \text{opt'} \]
Convex Relaxations

CSP

\[ \min_{x \in \Omega} f(x) \]

LP/SDP

\[ \min_{x \in \Omega'} f'(x) \]

\( \text{opt} \)

\( \text{opt'} \)
Constraint Satisfaction Problems

**IN:** set of variables, set of labels, set of constraints

**OUT:** assignment that satisfies and optimises constraints
Constraint Satisfaction Problems

**IN:** set of variables, set of labels, set of constraints

**OUT:** assignment that satisfies and optimises constraints
Constraint Satisfaction Problems

**IN:** set of variables, set of labels, set of constraints

**OUT:** assignment that *satisfies* and optimises constraints

---

3-Colour

![Graph Image](image-url)
Constraint Satisfaction Problems

**IN:** set of variables, set of labels, set of constraints

**OUT:** assignment that satisfies and optimises constraints

**3-Colour**

**Linear Equations over \(\mathbb{Z}_7\)**

\[
\begin{align*}
    x_1 + x_2 + x_3 &= 1 \\
    x_2 - x_4 + x_5 &= 0 \\
    x_1 - x_3 - x_4 &= 2
\end{align*}
\]
Constraint Satisfaction Problems

**IN:** set of variables, set of labels, set of constraints

**OUT:** assignment that satisfies and optimises constraints
Constraint Satisfaction Problems

**IN:** set of variables, set of labels, set of constraints

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\[(s,t)\text{-Min-Cut}\]
Constraint Satisfaction Problems

**IN:** set of variables, set of labels, set of constraints

**OUT:** assignment that satisfies and optimises constraints

(s,t)-Min-Cut

Vertex Cover
Constraint Satisfaction Problems

IN: set of variables, set of labels, set of constraints

OUT: assignment that satisfies and optimises constraints

- CSP = (V,D,C)
- CSP = homomorphism problem
- CSP = conjunctive query evaluation
- CSP = model checking of pp-fragment of FO
## Constraint Satisfaction Problems

### IN:
- set of variables, set of labels, set of constraints

### OUT:
- assignment that satisfies and optimises constraints

- **CSP = (V,D,C)**
- CSP = homomorphism problem
- CSP = conjunctive query evaluation
- CSP = model checking of pp-fragment of FO
Constraint Satisfaction Problems

**IN:** set of variables, set of labels, set of constraints

**OUT:** assignment that satisfies and optimises constraints

**Vertex Cover**

\[ I(x_1, \ldots, x_n) = \sum_{\{i,j\} \in E(G)} \psi(x_i, x_j) + \sum_{i \in V(G)} \tau(x_i) \]

\[ V = \{x_1, \ldots, x_n\}, \ D = \{0, 1\} \]
Constraint Satisfaction Problems

**IN:** set of variables, set of labels, set of constraints

**OUT:** assignment that satisfies and optimises constraints

**Vertex Cover**

\[ l(x_1, \ldots, x_n) = \sum_{\{i,j\} \in E(G)} \psi(x_i, x_j) + \sum_{i \in V(G)} \tau(x_i) \]

\[
\begin{array}{c|c|c}
 x & y & \psi(x, y) \\
 \hline
 1 & 1 & 0 \\
 1 & 0 & 0 \\
 0 & 1 & 0 \\
 0 & 0 & \infty
\end{array}
\]

\[
\begin{array}{c|c}
 x & \tau(x) \\
 \hline
 1 & 1 \\
 0 & 0
\end{array}
\]

\( V = \{x_1, \ldots, x_n\}, D = \{0, 1\} \)
Constraint Satisfaction Problems

**IN:** set of variables, set of labels, set of constraints

**OUT:** assignment that satisfies and optimises constraints

- CSP = (V,D,C)
- CSP = homomorphism problem
- CSP = conjunctive query evaluation
- CSP = model checking of pp-fragment of FO
Constraint Satisfaction Problems

**IN:** set of variables, set of labels, set of constraints

**OUT:** assignment that satisfies and optimises constraints

3-Colour

![Graph G](image1)

![Graph K₃](image2)

?
Constraint Satisfaction Problems

**IN:** set of variables, set of labels, set of constraints

**OUT:** assignment that satisfies and optimises constraints

**Vertex Cover**

![Graph G](image1)

![Graph B](image2)
Constraint Satisfaction Problems

**IN:** set of variables, set of labels, set of constraints

**OUT:** assignment that satisfies and optimises constraints

**k-Clique**

$K_k \rightarrow G$
Constraint Satisfaction Problems

**IN:** set of variables, set of labels, set of constraints

**OUT:** assignment that satisfies and optimises constraints

- CSP = (V,D,C)
- CSP = homomorphism problem
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Constraint Satisfaction Problems

**IN:** set of variables, set of labels, set of constraints

**OUT:** assignment that satisfies and optimises constraints

- CSP = (V,D,C)
- CSP = homomorphism problem
- CSP = conjunctive query evaluation
- CSP = model checking of pp-fragment of FO

**balance between generality and structure**
Convex Relaxations

CSP
\[ \min_{x \in \Omega} f(x) \]

LP/SDP
\[ \min_{x \in \Omega'} f'(x) \]

opt

opt'
Convex Relaxations

\[ \min_{x \in \Omega} f(x) \]

\[ \min_{x \in \Omega'} f'(x) \]

\[ \text{opt} = \text{opt}' \]
Vertex Cover

![Graph](image-url)
Vertex Cover

\[
\begin{align*}
\text{min} & \quad \sum_{v \in V} x_v \\
\text{s.t.} & \quad x_u + x_v \geq 1 \quad \forall (u, v) \in E \\
& \quad x_v \in \{0, 1\} \quad \forall v \in V
\end{align*}
\]
Vertex Cover

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\begin{align*}
\min & \quad \sum_{v \in V} x_v \\
\text{s.t.} & \quad x_u + x_v \geq 1 \quad \forall (u, v) \in E \\
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\end{align*}
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Thm: Basic LP solves CSPs with \((\mathbb{Q} \cup \{\infty\})\)-valued functions iff ...

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[Kolmogorov, Thapper, Ž. SICOMP’15]
Vertex Cover

\[ l(x_1, \ldots, x_n) = \sum_{\{i,j\} \in E(G)} \psi(x_i, x_j) + \sum_{i \in V(G)} \tau(x_i) \]

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\[ I(x_1, \ldots, x_n) = \sum_{\{i,j\} \in E(G)} \psi(x_i, x_j) + \sum_{i \in V(G)} \tau(x_i) \]

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  x & y & \psi(x, y) \\
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\end{array} \]

\[ \begin{array}{c|c}
  x & \tau(x) \\
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[Kolmogorov, Thapper, Ž. SICOMP’15]
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Vertex Cover is not solved by Basic LP!

[Kolmogorov, Thapper, Ž. SICOMP’15]
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Vertex Cover is **not** solved by Basic LP!

Submodular CSPs **solved** by Basic LP!

Kolmogorov, Thapper, Z. SICOMP’15
Vertex Cover

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Submodular CSPs solved by Basic LP!

[Kolmogorov, Thapper, Z. SICOMP'15]
Vertex Cover

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\begin{align*}
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Vertex Cover

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\text{min} & \quad \sum_{v \in V} x_v \\
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& \quad 0 \leq x_v \leq 1 \quad \forall v \in V
\end{align*}$$
Vertex Cover

\[
\begin{align*}
\text{min} & \quad \sum_{v \in V} \mathbb{P}_{S \sim \mathcal{D}\{\{v\}\}}[v \in S] \\
\text{s.t.} & \quad \mathcal{D}(U) \text{ is a distribution on } \mathcal{P}(U) \quad \forall U \subseteq V, |U| \leq k \\
& \quad \mathbb{P}_{S \sim \mathcal{D}\{u,v\}}[u \notin S, v \notin S] = 0 \quad \forall (u,v) \in E \\
& \quad \mathbb{P}_{S \sim \mathcal{D}(U')} = \mathbb{P}_{S \sim \mathcal{D}(U)} \quad \forall U' \subseteq U \subseteq V, |U| \leq k
\end{align*}
\]
**Thm:** Sherali-Adams\(k\) LP solves CSPs with \((\mathbb{Q} \cup \{\infty\})\)-valued functions iff …

[Thapper, Ž. SICOMP’17]
Vertex Cover

\[
\text{min } \sum_{v \in V} \mathbb{P}_{S \sim D\{\{v\}\}}[v \in S]
\]
\[
\text{s.t. } D(U) \text{ is a distribution on } \mathcal{P}(U) \quad \forall U \subseteq V, |U| \leq k
\]
\[
\mathbb{P}_{S \sim D\{\{u,v\}\}}[u \notin S, v \notin S] = 0 \quad \forall (u, v) \in E
\]
\[
\mathbb{P}_{S \sim D(U')} = \mathbb{P}_{S \sim D(U)} \quad \forall U' \subseteq U \subseteq V, |U| \leq k
\]

**Thm:** Sherali-Adams(k) LP solves CSPs with (\(\mathbb{Q} \cup \{\infty\}\))-valued functions iff …

(iif Sherali-Adams(3) solves)

[Thapper, Ž. SICOMP’17]
Vertex Cover

Thm: Sherali-Adams(k) LP solves CSPs with \((\mathbb{Q} \cup \{\infty\})\)-valued functions iff …

(iff Sherali-Adams(3) solves)

[Thapper, Ž. SICOMP’17]

Vertex Cover is not solved by Sherali-Adams(k) LP!
Vertex Cover

\[
\min \sum_{v \in V} \mathbb{P}_{S \sim D(\{v\})}[v \in S]
\]

s.t. \( D(U) \) is a distribution on \( \mathcal{P}(U) \)

\[
\mathbb{P}_{S \sim D(\{u,v\})}[u \notin S, v \notin S] = 0 \quad \forall (u,v) \in E
\]

\[
\mathbb{P}_{S \sim D(U')} = \mathbb{P}_{S \sim D(U)} \quad \forall U' \subseteq U \subseteq V, |U| \leq k
\]

\[\forall U \subseteq V, |U| \leq k\]
Vertex Cover

\[
\begin{align*}
\min & \quad \sum_{v \in V} \mathbb{P}_{S \sim D(\{v\})}[v \in S] \\
\text{s.t.} & \quad D(U) \text{ is a distribution on } \mathcal{P}(U) \quad \forall U \subseteq V, |U| \leq k \\
& \quad \mathbb{P}_{S \sim D(\{u,v\})}[u \notin S, v \notin S] = 0 \quad \forall (u,v) \in E \\
& \quad \mathbb{P}_{S \sim D(U')} = \mathbb{P}_{S \sim D(U)} \quad \forall U' \subseteq U \subseteq V, |U| \leq k
\end{align*}
\]

\[D \in \mathbb{R} \quad \Rightarrow \quad D \in \mathbb{R}^d\]
Vertex Cover

\[ \min \sum_{v \in V} \mathbb{P}_{S \sim D\{\{v\}\}}[v \in S] \]

s.t. 
\[ D(U) \text{ is a distribution on } \mathcal{P}(U) \]
\[ \mathbb{P}_{S \sim D\{\{u,v\}\}}[u \notin S, v \notin S] = 0 \]
\[ \mathbb{P}_{S \sim D(U')} = \mathbb{P}_{S \sim D(U)} \]
\[ \forall U \subseteq V, |U| \leq k \]
\[ \forall (u, v) \in E \]
\[ \forall U' \subseteq U \subseteq V, |U| \leq k \]
Vertex Cover

Thm: Sum-of-Squares SDP solves CSPs with $(\mathbb{Q} \cup \{\infty\})$-valued functions iff …

[Thapper, Ž. ACM ToCT'18]
Vertex Cover

**Thm:** Sum-of-Squares SDP solves CSPs with \((\mathbb{Q}_\cup\{\infty\})\)-valued functions iff …

(Sherali-Adams(3) or not even Sum-of-Squares\((\frac{n}{1024})\)) \[\text{[Thapper, Ž. ACM ToCT'18]}\]
**Vertex Cover**

**Thm:** Sum-of-Squares SDP solves CSPs with \(\mathbb{Q} \cup \{\infty\}\)-valued functions iff …

(Sherali-Adams(3) or not even Sum-of-Squares(\(\frac{n}{1024}\)))

[Thapper, Ž. ACM ToCT’18]

Vertex Cover is not solved by any poly-size SDP relaxation!

[Lee, Raghavendra, Steurer STOC’15]
Exact Solvability

Power of Basic LP  
[ Kolmogorov, Thapper, Ž. SICOMP’15 ]

Power of Sherali-Adams LP  
[ Thapper, Ž. SICOMP’17 ]

Power of Sum-of-Squares SDP  
[ Thapper, Ž. ACM ToCT’18 ]
Exact Solvability

Power of Basic LP

Power of Sherali-Adams LP

Power of Sum-of-Squares SDP

Fixed RHS: \texttt{CSP(-,B)}
Exact Solvability

Power of Basic LP [Kolmogorov, Thapper, Ž. SICOMP’15]

Power of Sherali-Adams LP [Thapper, Ž. SICOMP’17]

Power of Sum-of-Squares SDP [Thapper, Ž. ACM ToCT’18]

Fixed RHS: CSP(-,B)
Exact Solvability

Power of Basic LP

Power of Sherali-Adams LP

Power of Sum-of-Squares SDP

Fixed RHS: CSP(-,B)

Fixed LHS: CSP(A,-)

[Kolmogorov, Thapper, Ž. SICOMP'15]

[Thapper, Ž. SICOMP'17]

[Thapper, Ž. ACM ToCT’18]
CSP($\mathcal{A}, -$)
\[ \text{CSP}(\mathcal{A},-) \]

\[ \text{k-Clique} = \text{CSP}(\mathcal{A},-) \text{ for } \mathcal{A} = \{K_k\}_{k \geq 1} \]
CSP($\mathcal{A}, -$)
CSP(\(\mathcal{A},-\)) in PTIME if \(\text{tw}(\mathcal{A}) < \infty\)

[Freuder JACM’82, AAAI’90]
CSP(\(A, -\))
CSP(\(\mathcal{A},-\)) in PTIME if \(\text{tw}(\text{core}(\mathcal{A})) < \infty\)  

[Dalmau, Kolaitis, Vardi CP'02]
CSP($\mathcal{A},-$) in PTIME if tw(core($\mathcal{A}$)) < $\infty$
CSP($\mathcal{A},-$) not in PTIME otherwise

[Dalmau, Kolaitis, Vardi CP’02]
[Grohe JACM’07]
CSP(𝒜, -) with \((\mathbb{Q} \cup \{\infty\})\)-valued fns
CSP(\(\mathcal{A},-\)) with \((\mathbb{Q} \cup \{\infty\})\)-valued fns
CSP($\mathcal{A}, -$) with $(\mathbb{Q} \cup \{\infty\})$-valued fns

$\text{tw}(\text{core}(\text{pos}(\mathcal{A}))) < \infty$

$\text{tw}(\mathcal{A}) < \infty$

PTIME

W[1]-hard
$\text{CSP}(\mathcal{A}, -)$
CSP($\mathcal{A}, -$) in PTIME but $tw(\mathcal{A}) = \infty$
CSP(\(\mathcal{A},-\))
CSP($\mathcal{A}, -$) not in PTIME but $\text{tw}(\text{pos}(\mathcal{A})) < \infty$
CSP($\mathcal{A}, -$) with $(\mathbb{Q} \cup \{\infty\})$-valued fns

$\text{tw}(\text{core}(\text{pos}(\mathcal{A}))) < \infty$

$\text{tw}(\mathcal{A}) < \infty$

$\text{PTIME}$

$\text{W}[1]$-hard
CSP(\(\mathcal{A},-\)) with \((\mathbb{Q} \cup \{\infty\})\)-valued fns

\[
\text{tw}(\text{core}(\text{pos}(\mathcal{A}))) < \infty
\]

\[
\text{tw}(\text{vcore}(\mathcal{A})) < \infty
\]

\[
\text{tw}(\mathcal{A}) < \infty
\]

\(\text{PTIME}\)

\(\text{W}[1]\)-hard
Theorem [Carbonnel, Romero, Ž. SICOMP’21+]

Assume \( \text{FPT} \neq \text{W}[1] \). Let \( \mathcal{A} \) be a recursively enumerable class of \((\mathbb{Q} \cup \{\infty\})\)-valued structures of bounded arity. TFAE:

1. \( \text{CSP}(\mathcal{A},-) \in \text{PTIME} \)
2. \( p\text{-CSP}(\mathcal{A},-) \in \text{FPT} \)
3. \( \mathcal{A} \) is bounded treewidth modulo valued equivalence

\[
\text{tw}(\text{vcore}(\mathcal{A})) < \infty
\]

\[
\text{tw}(\mathcal{A}) < \infty
\]

\[
(\lvert A \rvert + \lvert B \rvert)^O(1)
\]

\[
f(\lvert A \rvert) \cdot \lvert B \rvert^O(1)
\]
Theorem [Carbonnel, Romero, Ž. SICOMP’21+]

Let $\mathcal{A}$ be a $(\mathbb{Q} \cup \{\infty\})$-valued $\sigma$-structure.
Let $k \geq \text{max-arity}(\sigma)$.
Let $\mathcal{A}'$ be a valued core of $\mathcal{A}$.

The $k$th level of Sherali-Adams is exact

$$opt_k(\mathcal{A}, \mathcal{B}) = opt(\mathcal{A}, \mathcal{B})$$ for every valued $\sigma$-structure $\mathcal{B}$

$\uparrow$

1. $tw(\mathcal{A}') \leq k - 1$
Theorem [Carbonnel, Romero, Ž. SICOMP’21+]

Let $A$ be a $(\mathbb{Q} \cup \{\infty\})$-valued $\sigma$-structure.
Let $k \geq 1$.
Let $A'$ be a valued core of $A$.

$k$th level of Sherali-Adams is exact

\[ \text{opt}_k(A, B) = \text{opt}(A, B) \text{ for every valued } \sigma\text{-structure } B \]

\[ \updownarrow \]

1. $\text{tw}_{ms}(A') \leq k - 1$
2. overlap of $A'$ is $\leq k$
Exact Solvability

Power of Basic LP  \[\text{[Kolmogorov, Thapper, Ž. SICOMP'15]}\]

Power of Sherali-Adams LP  \[\text{[Thapper, Ž. SICOMP'17]}\]

Power of Sum-of-Squares SDP  \[\text{[Thapper, Ž. ACM ToCT'18]}\]

Fixed RHS: CSP(-,B)

Fixed LHS: CSP(A,-)

Power of Sherali-Adams LP  \[\text{[Carbonnel, Migero, Ž. SICOMP'21+]}\]
How to Relax

“I can’t find an efficient algorithm, but neither can all these famous people.”
How to Relax

“I can’t find an efficient algorithm, but neither can all these famous people.”

Satisfy only a fraction of the constraints!
Max-Cut
Max-Cut

planar PTIME

[Hadlock SICOMP'75]
Max-Cut

planar PTIME

excluded-minor PTAS

[Hadlock SICOMP’75]

[Grohe Combinatorica’03, Demaine et al. FOCS’05]
Max-Cut

planar PTIME
excluded-minor PTAS
dense PTAS

[Hadlock SICOMP’75]

[Grohe Combinatorica’03, Demaine et al. FOCS’05]

[Arora et al. STOC’95, Frieze & Kannan FOCS’96]
Max-Cut

planar PTIME
excluded-minor PTAS
dense PTAS

[Hadlock SICOMP’75]

[Grohe Combinatorica’03, Demaine et al. FOCS’05]

[Arora et al. STOC’95, Frieze & Kannan FOCS’96]

How do you unify sparse and dense graph classes?
Max-Cut

planar PTIME
excluded-minor PTAS
dense PTAS

How do you unify sparse and dense graph classes?

**Thm:** PTAS for Max-CSPs that are treewidth-pliable.  
[Romero, Wrochna, Ž. SODA’21]
Max-Cut

planar PTIME
excluded-minor PTAS
dense PTAS

Thm: PTAS for Max-CSPs that are treewidth-pliable. [Romero, Wrochna, Ž. SODA’21]
Max-Cut

fr-tw-fragility

hyperfiniteness

planar PTIME

excluded-minor PTAS

dense PTAS

Thm: PTAS for Max-CSPs that are treewidth-pliable.

[Romero, Wrochna, Ž. SODA’21]

How do you unify sparse and dense graph classes?

[Hadlock SICOMP’75]

[Grohe Combinatorica’03, Demaine et al. FOCS’05]

[Arora et al. STOC’95, Frieze & Kannan FOCS’96]
Thm: PTAS for Max-CSPs that are treewidth-pliable.

[Romero, Wrochna, Ž. SODA’21]
Max-Cut

Thm: PTAS for Max-CSPs that are treewidth- pliable. [Romero, Wrochna, Ž. SODA’21]

\[ \forall \varepsilon > 0 \exists k \in \mathbb{N} \forall G \in \mathcal{C} \exists G' \text{ with } tw(G') \leq k \land d_{opt}(G, G') \leq \varepsilon \]
Max-Cut

planar PTIME
excluded-minor PTAS
dense PTAS

fr-tw-fragility
hyperfiniteness
regularity partitions

How do you unify sparse and dense graph classes?

Thm: PTAS for Max-CSPs that are treewidth-pliant.

∀ε > 0 ∃k ∈ ℕ ∀G ∈ ℂ ∃G′ with \( \text{tw}(G') \leq k \) ∧ \( d_{opt}(G, G') \leq \varepsilon \)

\( d_{opt}(G, G') = \inf_{\varepsilon} \text{Max-Cut}(G) = (1 \pm \varepsilon)\text{Max-Cut}(G') \)

[Hadlock SICOMP’75]
[Grohe Combinatorica’03, Demaine et al. FOCS’05]
[Arora et al. STOC’95, Frieze & Kannan FOCS’96]

[Romero, Wrochna, Ž. SODA’21]
How to Relax

“I can’t find an efficient algorithm, but neither can all these famous people.”
How to Relax

“I can’t find an efficient algorithm, but neither can all these famous people.”

Satisfy a relaxed version of the constraints!
Approximate Colouring
Approximate Colouring

(k,c)-colour

find a c-colouring of a k-colourable graph
Approximate Colouring

Conjecture: for any constant $3 \leq k \leq c$, $(k,c)$-colour is NP-hard.
Approximate Colouring

Known hardness:
Approximate Colouring

Known hardness:

$(k, k)$ for $k \geq 3$

[Karp CCC'72]
Approximate Colouring

**Known hardness:**

\[(k, k) \text{ for } k \geq 3\]

\[(k, 2k - 5) \text{ for } k \geq 6\]

\[(k, k + 2\lfloor \frac{k}{3} \rfloor - 1) \text{ for } k \geq 3\]

\[(k, 2k - 2) \text{ for } k \geq 3\]

[Karp CCC’72]

[Garey & Johnson JACM’76]

[Khanna et al. Combinatorica’00]

[Brakensiek & Guruswami CCC’16]
Approximate Colouring

**Known hardness:**

\[(k, k) \text{ for } k \geq 3\]  \[\text{[Karp CCC’72]}\]

\[(k, 2k - 5) \text{ for } k \geq 6\]  \[\text{[Garey & Johnson JACM’76]}\]

\[(k, k + 2 \lfloor \frac{k}{3} \rfloor - 1) \text{ for } k \geq 3\]  \[\text{[Khanna et al. Combinatorica’00]}\]

\[(k, 2k - 2) \text{ for } k \geq 3\]  \[\text{[Brakensiek & Guruswami CCC’16]}\]

\[(k, 2k - 1) \text{ for } k \geq 3\]  \[\text{[Bulín et al. STOC’19]}\]
Approximate Colouring

**Known hardness:**

- \((k, k)\) for \(k \geq 3\) [Karp CCC'72]
- \((k, 2k - 5)\) for \(k \geq 6\) [Garey & Johnson JACM'76]
- \((k, k + 2\lfloor \frac{k}{3}\rfloor - 1)\) for \(k \geq 3\) [Khanna et al. Combinatorica'00]
- \((k, 2k - 2)\) for \(k \geq 3\) [Brakensiek & Guruswami CCC'16]
- \((k, 2k - 1)\) for \(k \geq 3\) [Bulín et al. STOC'19]

**Thm:** \((k, \left(\frac{k}{\lfloor k/2 \rfloor}\right) - 1)\) for \(k \geq 4\). [Wrochna, Ž. SODA'20]
Approximate Colouring

Known hardness:

\[(k, k) \text{ for } k \geq 3\]  \[[\text{Karp CCC'72}]\]

\[(k, 2k - 5) \text{ for } k \geq 6\]  \[[\text{Garey & Johnson JACM'76}]\]

\[(k, k + 2\lfloor\frac{k}{3}\rfloor - 1) \text{ for } k \geq 3\]  \[[\text{Khanna et al. Combinatorica'00}]\]

\[(k, 2k - 2) \text{ for } k \geq 3\]  \[[\text{Brakensiek & Guruswami CCC'16}]\]

\[(k, 2k - 1) \text{ for } k \geq 3\]  \[[\text{Bulin et al. STOC'19}]\]

**Thm:**  \[(k, \left(\frac{k}{\lfloor k/2 \rfloor}\right) - 1) \text{ for } k \geq 4.\]  \[[\text{Wrochna, Ž. SODA'20}]\]

Finding a c-colouring a 3-colourable graph believed to be hard!

(but only known for \(3 \leq c \leq 5\))
Approximate Graph Homomorphism

Finding a 3-colouring of a 2-colourable graph easy!
Approximate Graph Homomorphism

Finding a 3-colouring of a 2-colourable graph easy!

**Thm:** Finding a 3-colouring of $(2+\varepsilon)$-colourable graph is NP-hard.

[Krokhin, Opršal, Wrochna, Ž. ‘21+]
Approximate Graph Homomorphism

Finding a 3-colouring of a 2-colourable graph easy!

**Thm:** Finding a 3-colouring of \((2+\varepsilon)\)-colourable graph is NP-hard.

\[G \rightarrow \begin{array}{c}
\text{Graph}
\end{array} \rightarrow \begin{array}{c}
\text{Graph}
\end{array} \rightarrow \begin{array}{c}
\text{Graph}
\end{array}\]

[Krokhin, Opršal, Wrochna, Ž. ‘21+]
Approximate Graph Homomorphism

Finding a 3-colouring of a 2-colourable graph easy!

**Thm:** Finding a 3-colouring of \((2+\varepsilon)\)-colourable graph is NP-hard.

[Krokhin, Opršal, Wrochna, Ž. ‘21+]

\[ G \rightarrow \rightarrow \rightarrow \]

every non-bipartite 3-colourable \( H \)
Approximate Graph Homomorphism

Finding a 3-colouring of a 2-colourable graph easy!

**Thm:** Finding a 3-colouring of $(2+\varepsilon)$-colourable graph is NP-hard.

4-$\varepsilon$ every non-bipartite 3-colourable H

[Krokhin, Opršal, Wrochna, Ž. ‘21+]
Approximate Graph Homomorphism

4-$\varepsilon$

Finding a 3-colouring of a 2-colourable graph easy!

**Thm:** Finding a 3-colouring of $(2+\varepsilon)$-colourable graph is NP-hard.

[Krokhin, Opršal, Wrochna, Ž. ‘21 +]

\[
G \rightarrow \quad \text{every non-bipartite 3-colourable H} \quad \rightarrow \quad \text{topology matters!}
\]
Approximate Graph Homomorphism

4-ε Finding a 3-colouring of a 2-colourable graph easy!

**Thm:** Finding a 3-colouring of \((2+\varepsilon)\)-colourable graph is NP-hard.

Thank you!

[Approximate Graph Homomorphism]

Finding a 3-colouring of a 2-colourable graph easy!

**Thm:** Finding a 3-colouring of \((2+\varepsilon)\)-colourable graph is NP-hard.

[Krokhin, Opršal, Wrochna, Ž. ‘21+]

\(G \rightarrow \) every non-bipartite 3-colourable \(H\)  

topology matters!

Thank you!