Power of Algorithms in Discrete Optimisation

Stanislav Živný (Oxford)

25th November 2021

What captures efficient computation?

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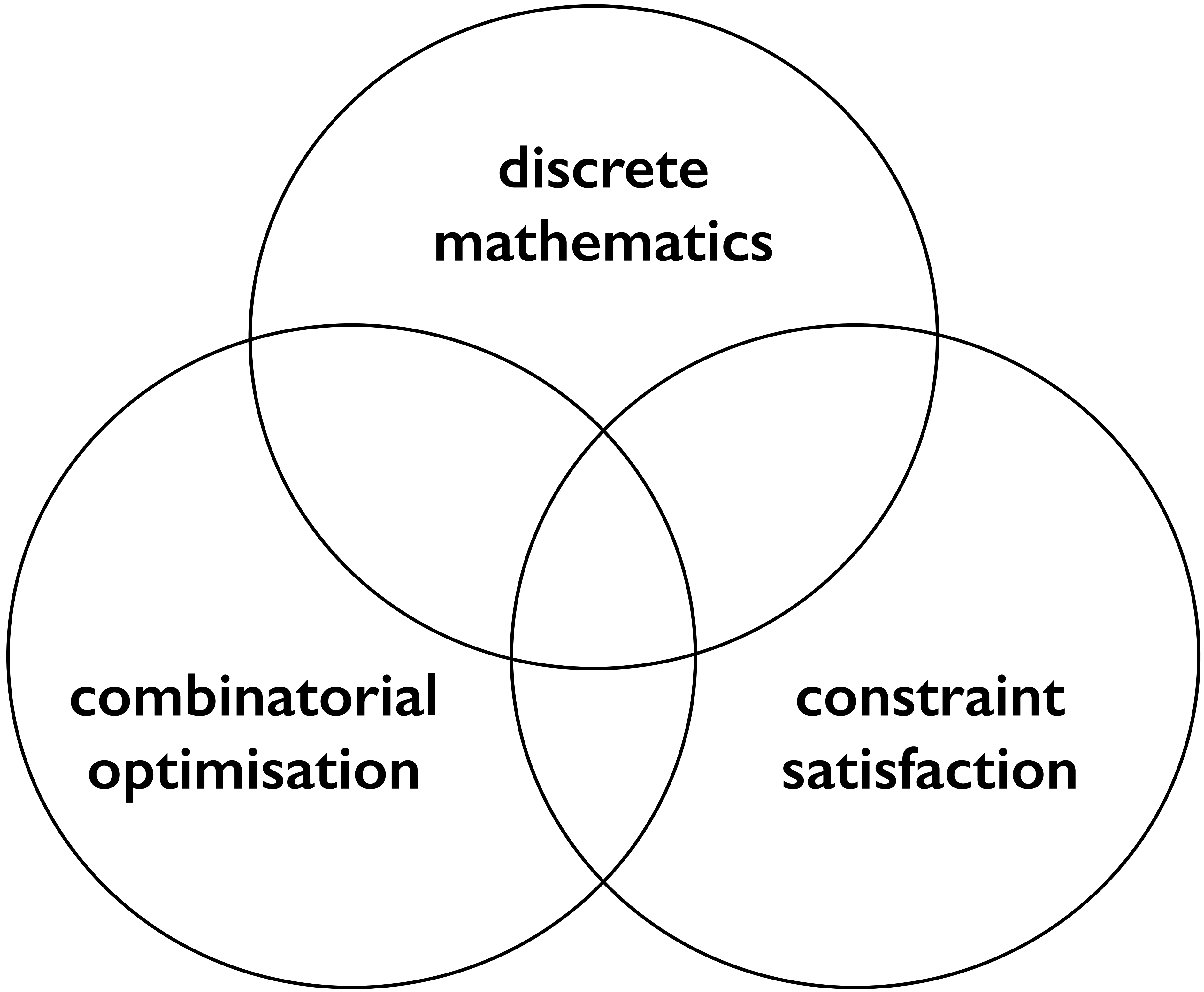
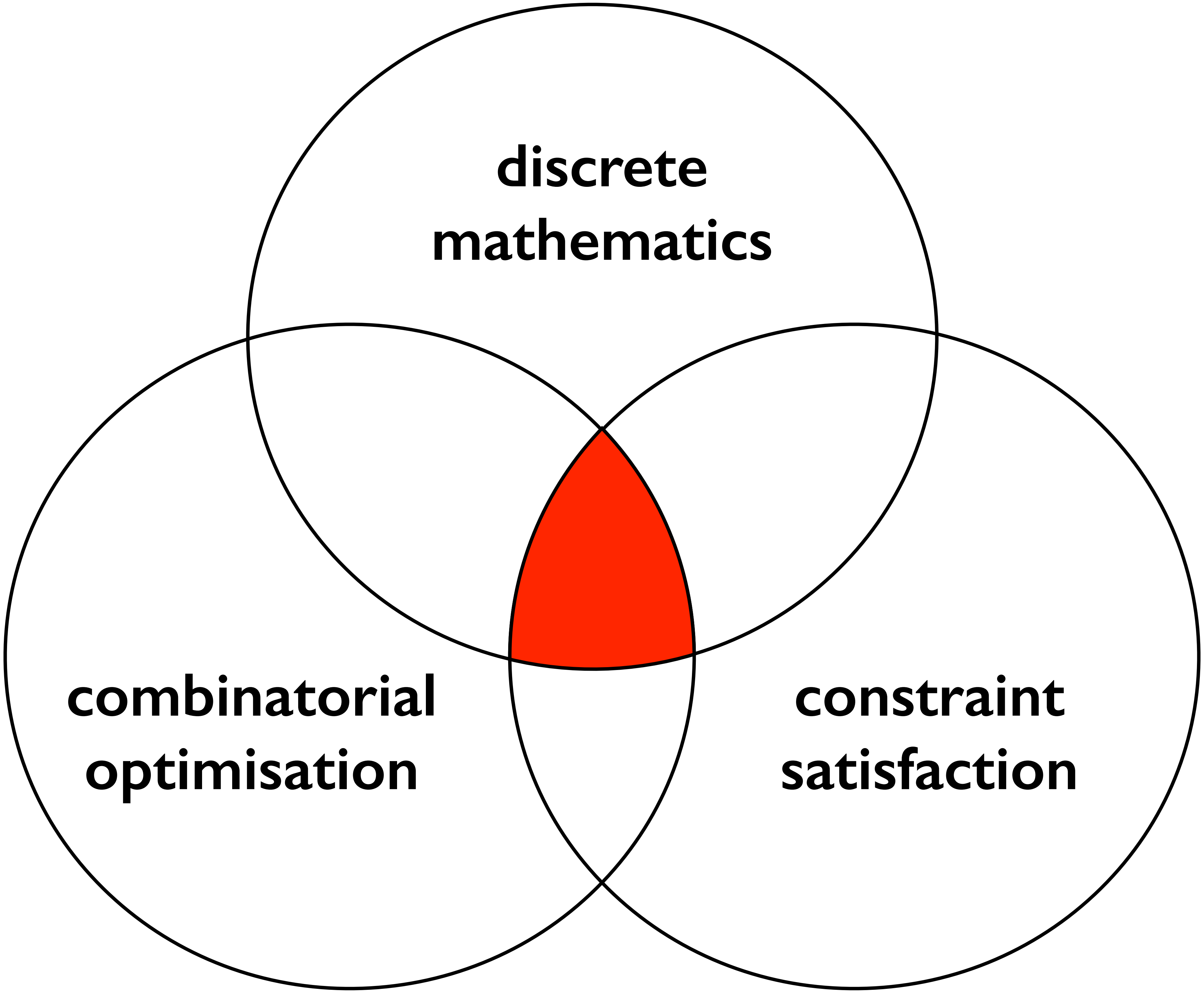
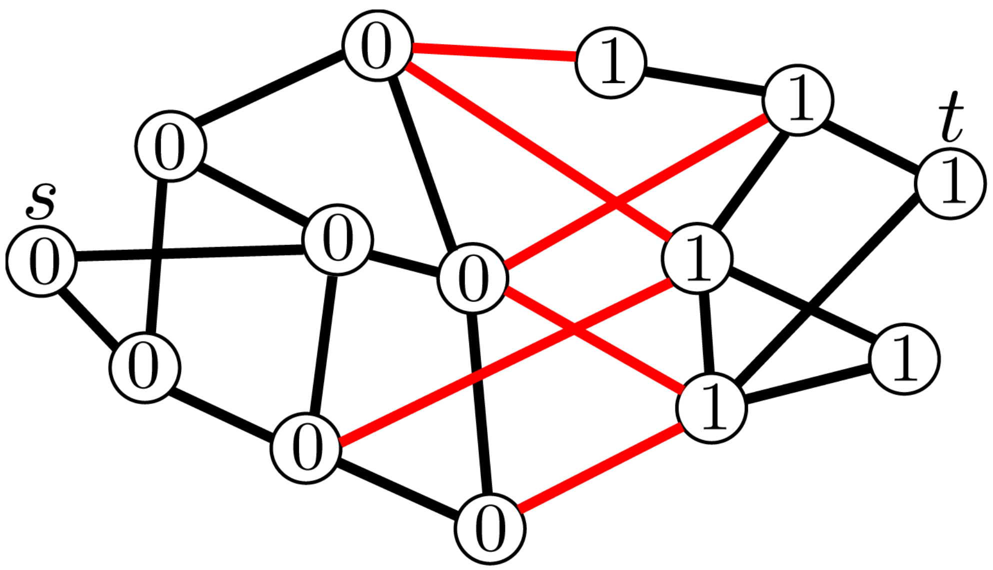
What are the limits of efficient algorithms?


discrete
mathematics
combinatorial
optimisation
constraint
satisfaction

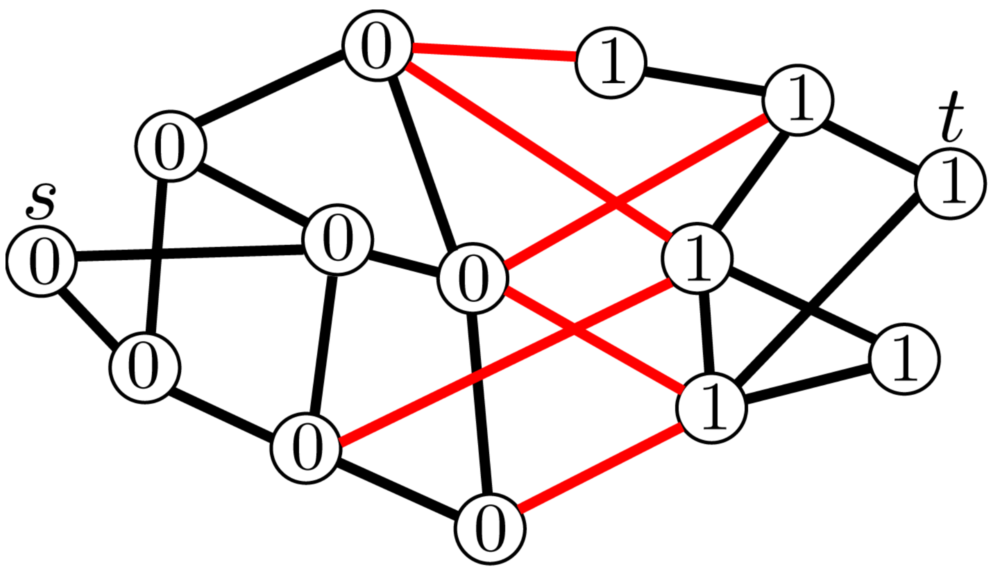


convex relaxations

(s,t)-Min-Cut



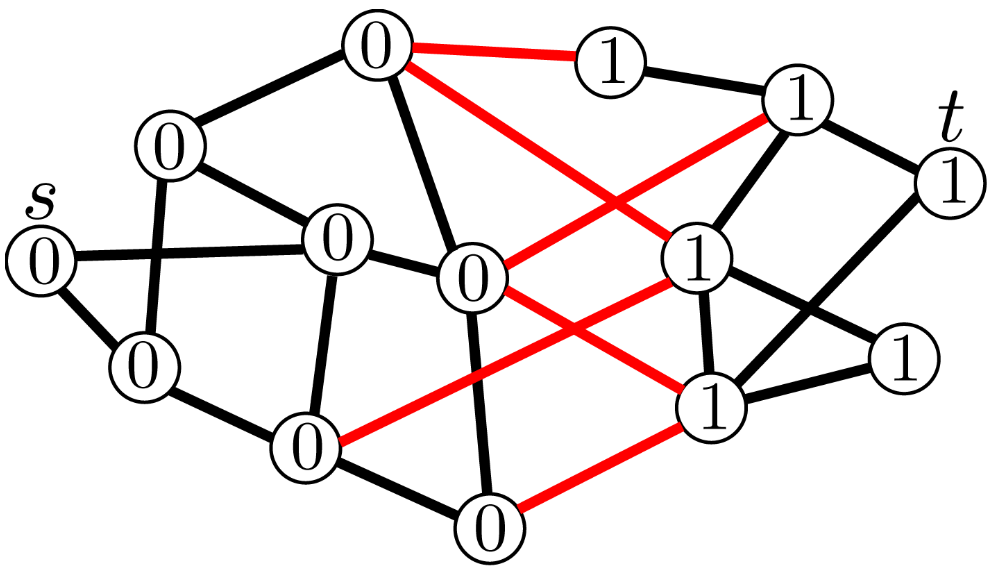
(s,t)-Min-Cut



$$\min_{\substack{x_1,\ldots,x_n \in \{0,1\} \\ s=0, t=1}} \sum_{\{i,j\} \in E(G)} \phi(x_i, x_j)$$

(s,t)-Min-Cut

x	y	$\phi(x,y)$
0	0	0
0	1	1
1	0	1
1	1	0



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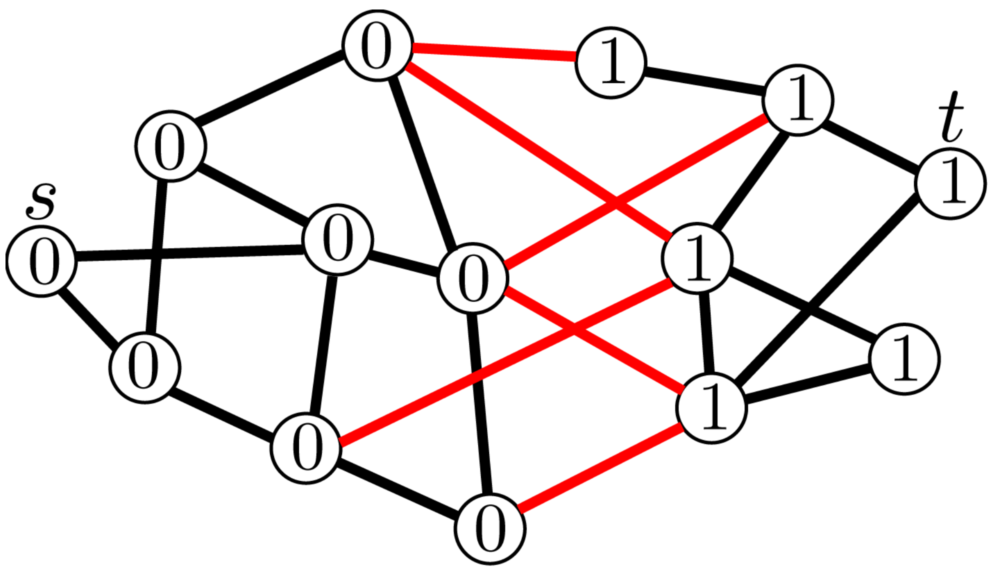

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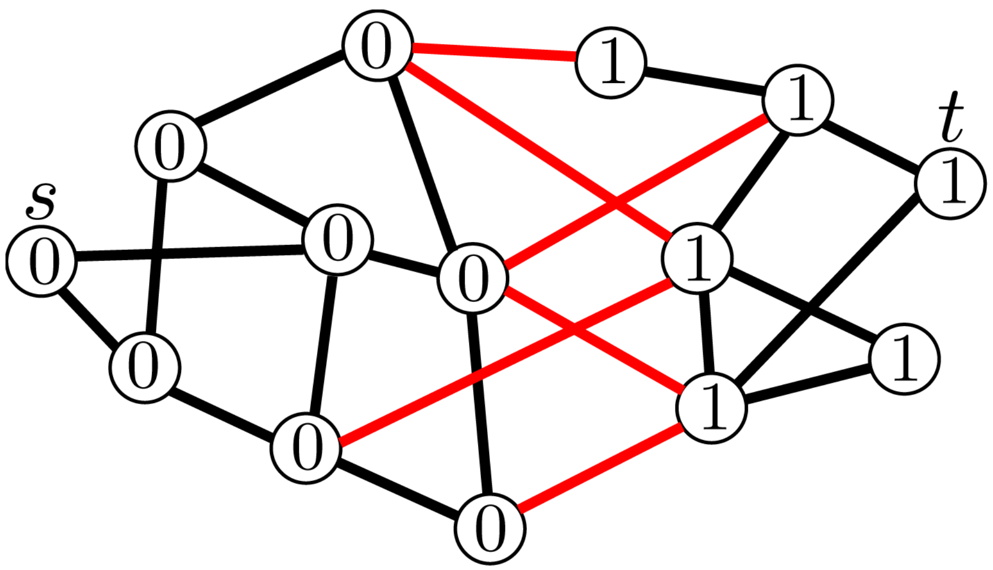

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submodularity

(s,t)-Min-Cut

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- symmetric tournament pair [*Cohen et al. TCS*'08]
- submodularity on binary trees [*Kolmogorov MFCS*'11]
- submodularity on distributive lattices [*Iwata et al. JACM*'01]
- submodularity on dimaonds [*Kuivinen DO*'11]
- skew bisubmodularity [*Huber et al. SODA*'13/*SICOMP*'14]
- bisubmodularity on 3-element domains [*Fujishige & Iwata SIDMA*'05]
- binary functions on unbounded domains [*Cohen et al. JAIR*'04]
- binary weighted functions [*Cohen et al. DAM*'05]
- {0,1}-valued functions on $|D| = 2$ [*Creignou JCSS*'95]
- {0,1}-valued functions on $|D| = 3$ [*Jonsson et al. SICOMP*'06]
- {0,1}-valued functions on $|D| = 4$ [*Jonsson et al. CP*'11]
- {0,1}-valued conservative functions [*Deineko et al. JACM*'08]
- functions on $|D| = 2$ [*Cohen et al. AIJ*'06]
- functions on $|D| = 3$ [*Huber et al. SODA*'13]
- weighted colourings [*Jonsson & Krokhin JCSS*'07]
- binary BLP [*Werner PAMI*'07, *PAMI*'10]
- conservative functions [*Kolmogorov & Ž. JACM*'13]
- min 0-extension problems [*Hirai SODA*'13]

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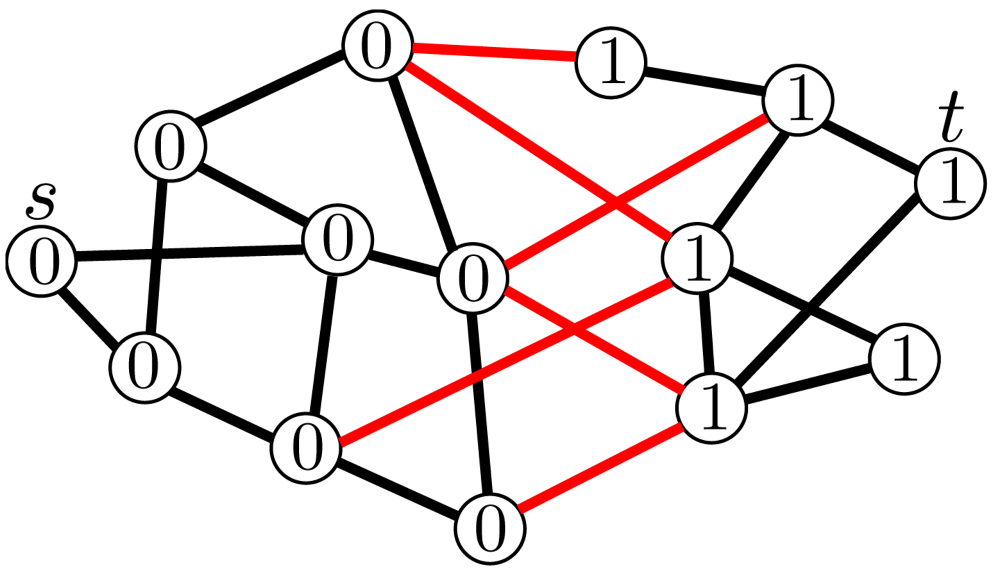

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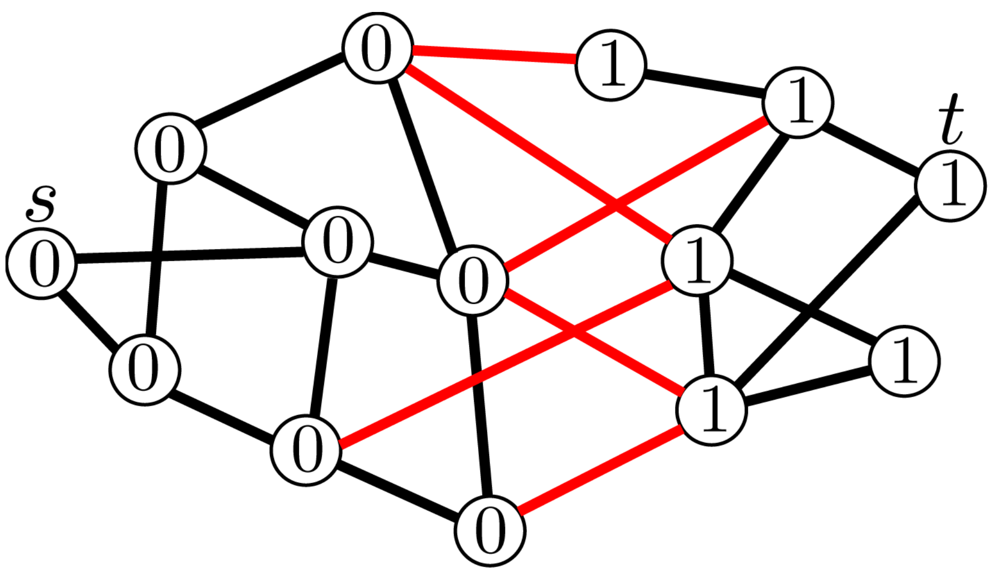

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Thm: CSPs with $\mathbb{Q}$-valued fns in PTIME if …, NP-complete otherwise

[Thapper, Ž. JACM'16]

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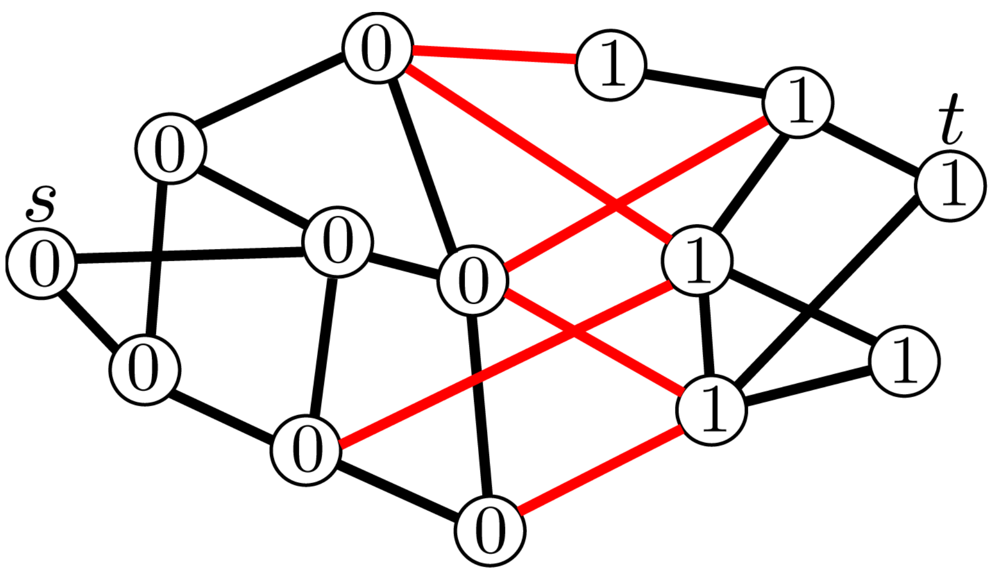

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dichotomy!

Thm: CSPs with ℚ-valued fns in PTIME if …, NP-complete otherwise

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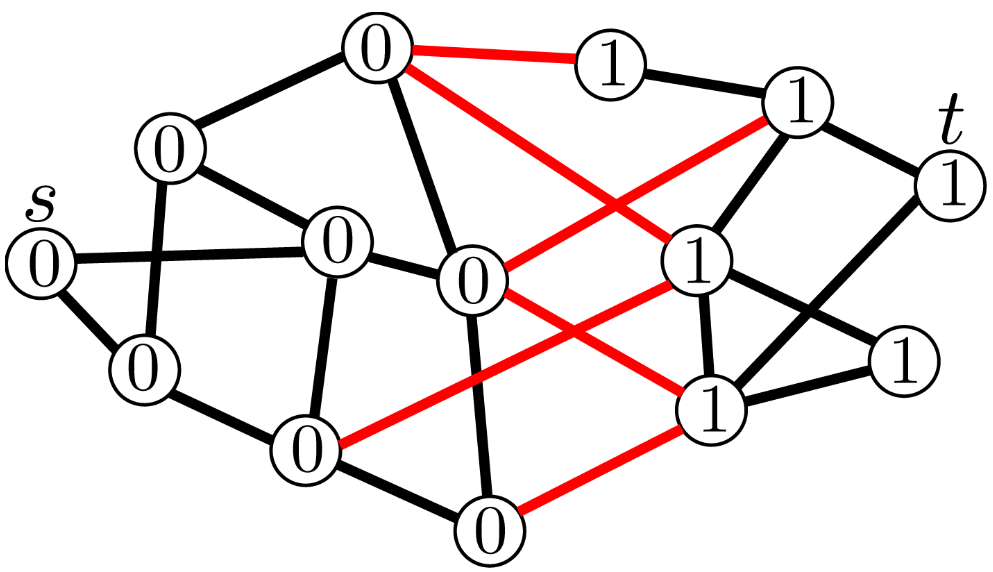

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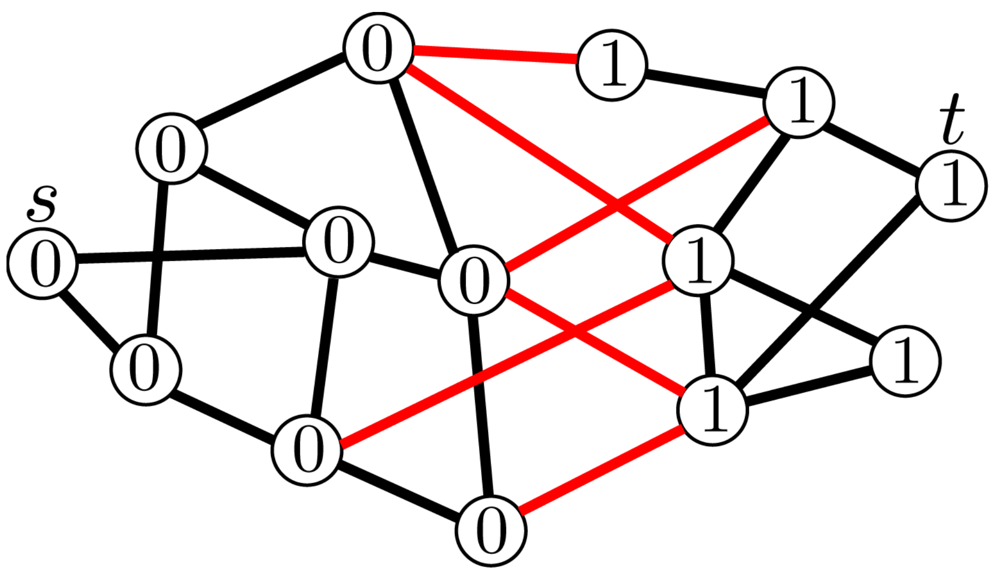

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submodularity

dichotomy!

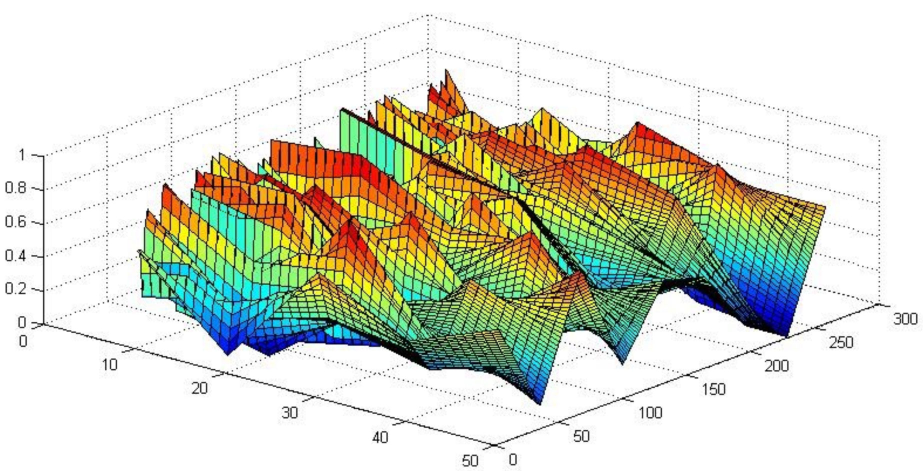
Thm: CSPs with $\mathbb{Q}$-valued fns in PTIME if …, NP-complete otherwise
(all tractable cases via Basic LP)

*[Thapper, **Ž**. JACM'16]*

Convex Relaxations

Convex Relaxations

$$\min_{\mathbf{x} \in \Omega} f(\mathbf{x})$$



Convex Relaxations

$$\min_{\mathbf{x}\in\Omega} f(\mathbf{x})$$

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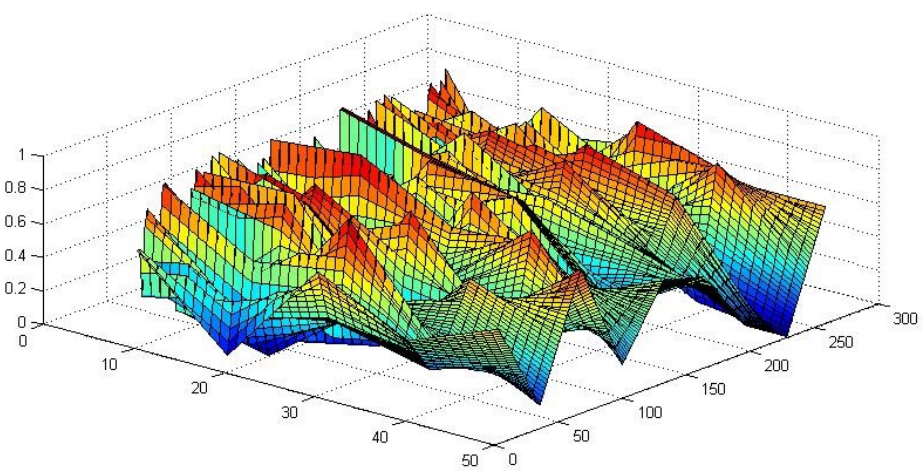

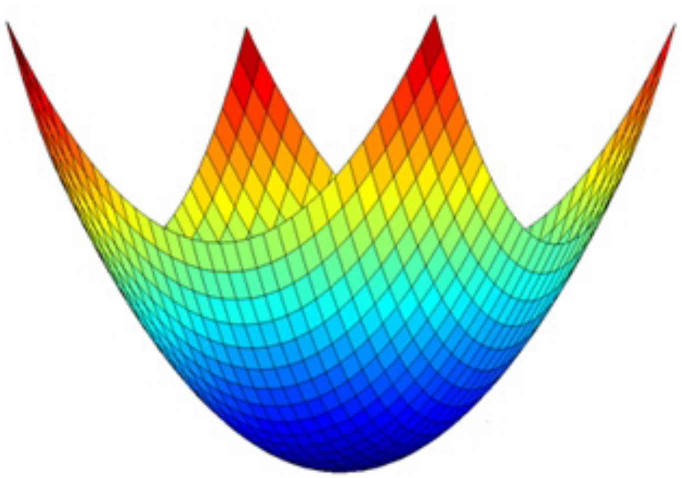

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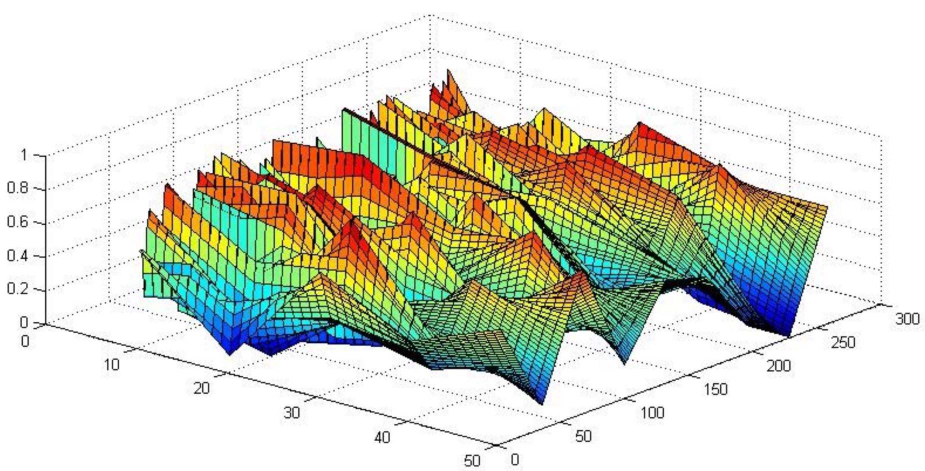

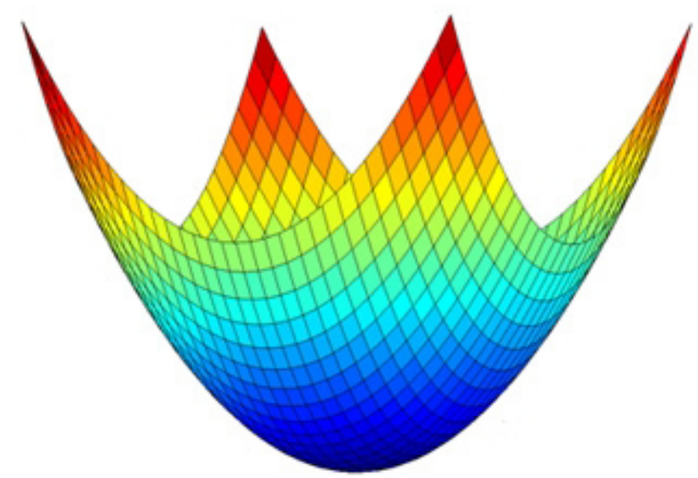

opt'

Convex Relaxations

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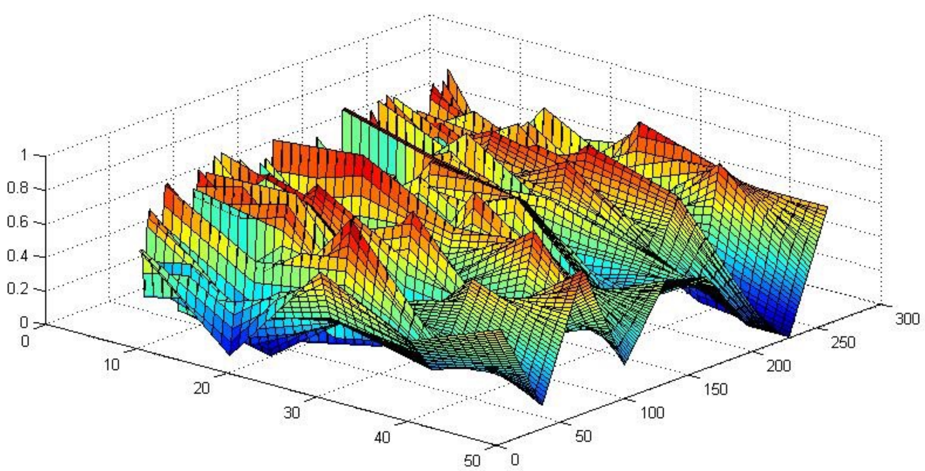
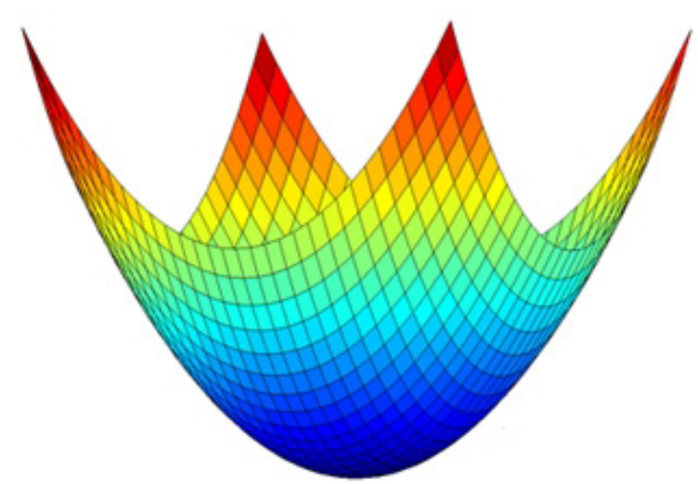
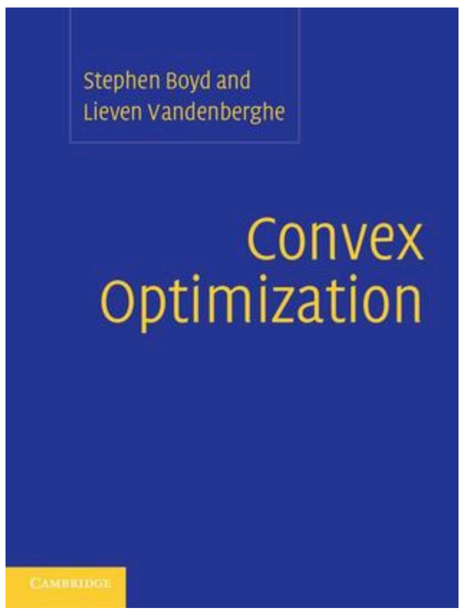


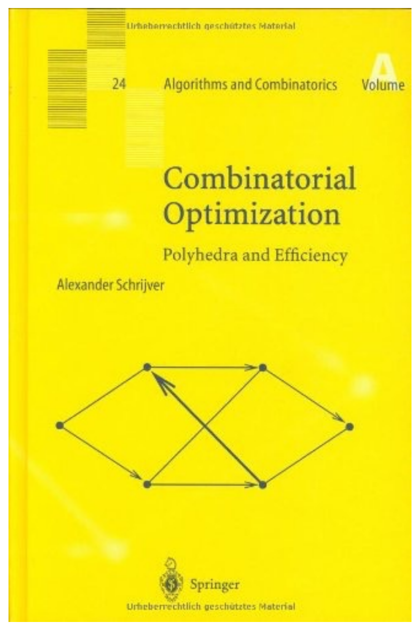


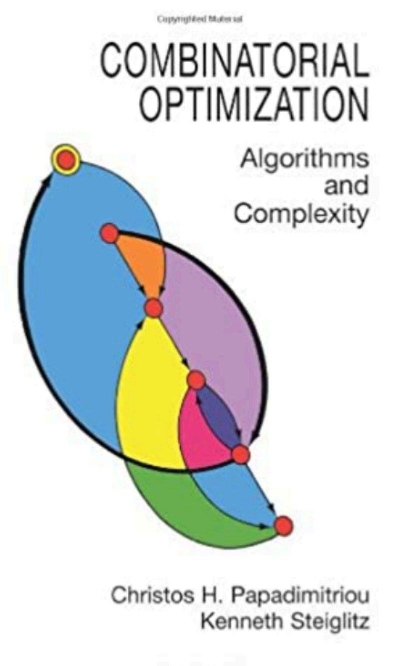
$opt \qquad opt'$

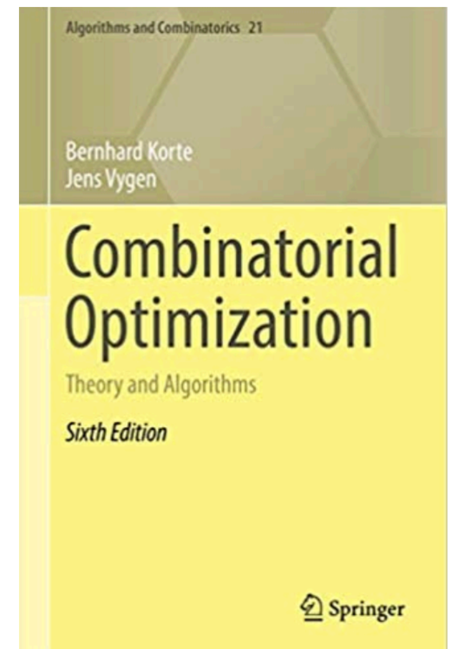
Convex Relaxations

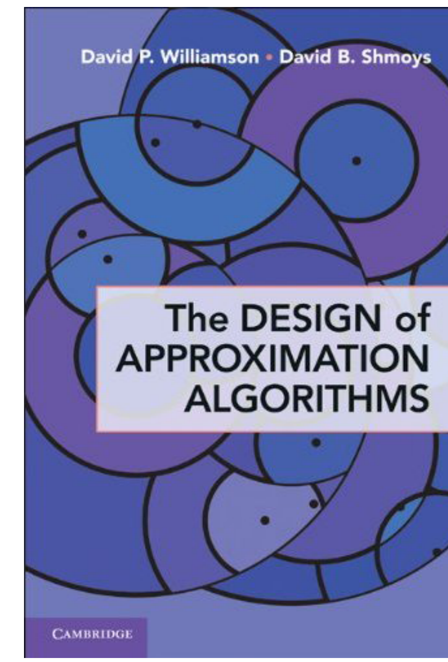
Linear (**LP**) and Semidefinite (**SDP**) Programming

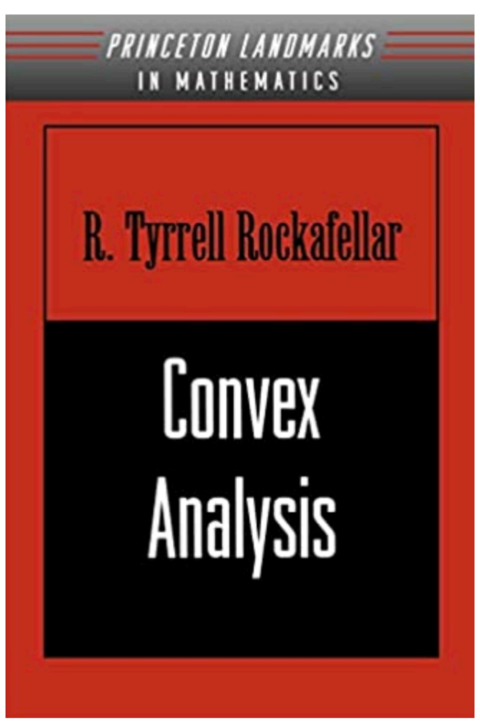


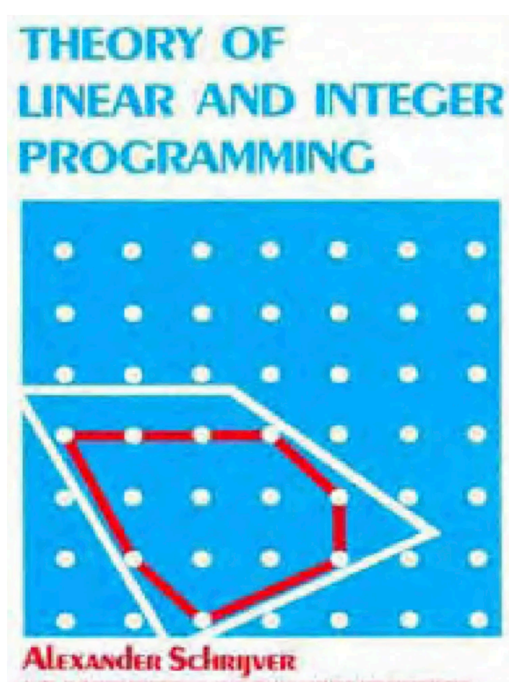


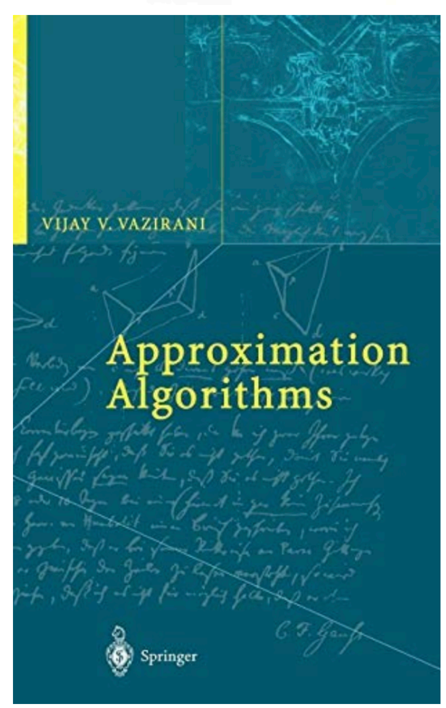


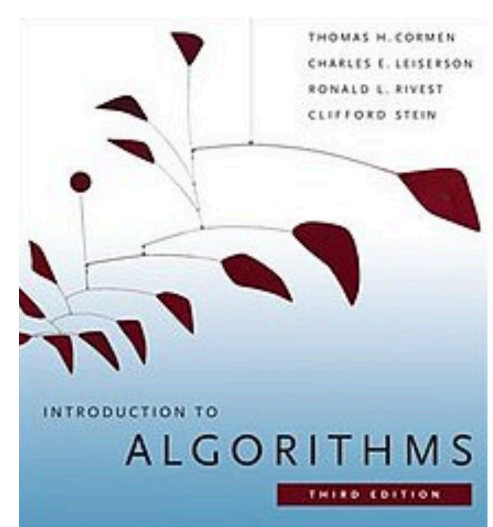


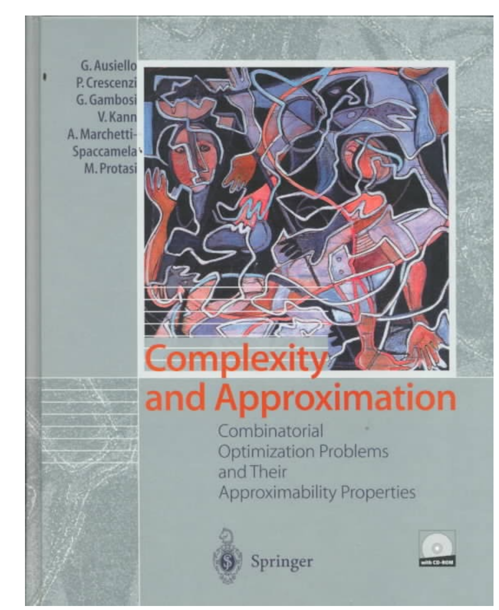








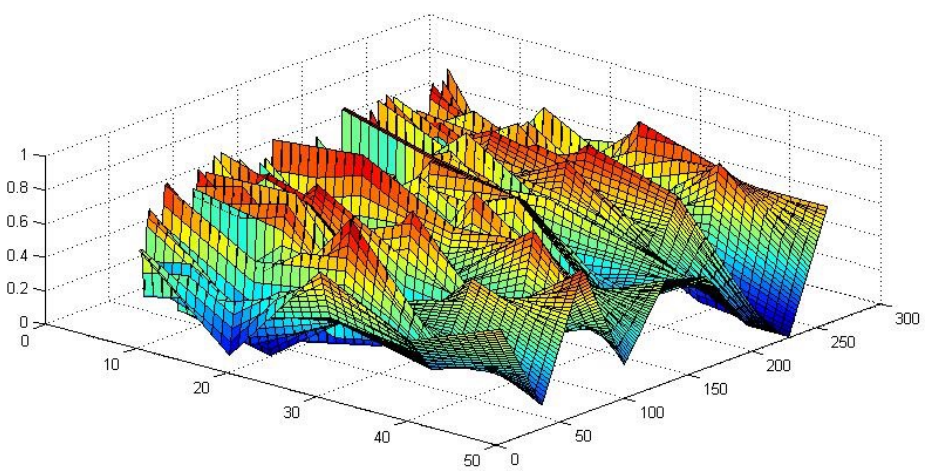


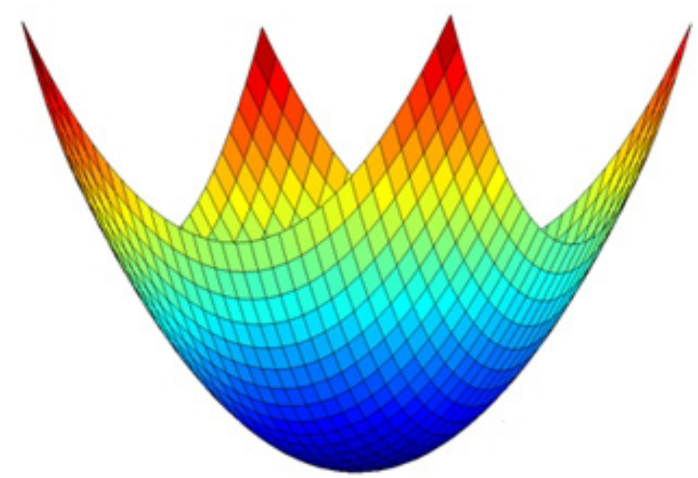


Convex Relaxations

LP/SDP

$$\min_{\mathbf{x}\in\Omega} f(\mathbf{x}) \qquad \longrightarrow \qquad \min_{\mathbf{x}\in\Omega'} f'(\mathbf{x})$$



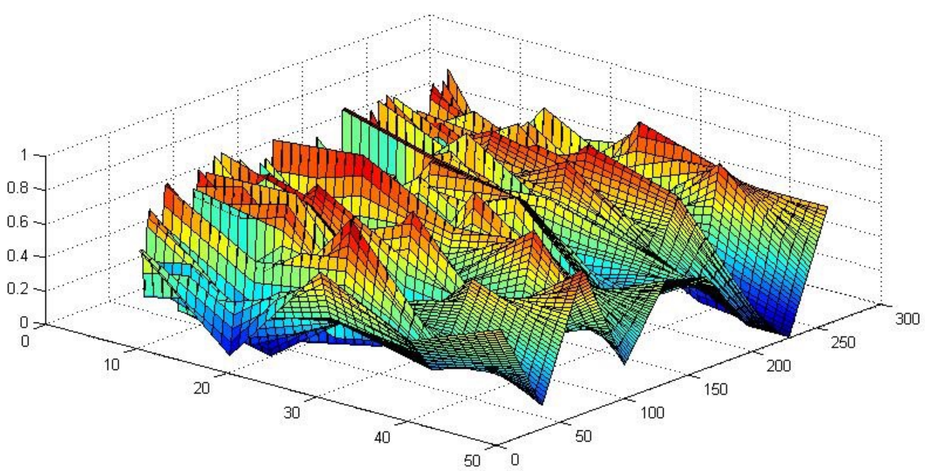


$$opt \qquad \longleftarrow \qquad opt'$$

Convex Relaxations

CSP

$$\min_{\mathbf{x}\in\Omega} f(\mathbf{x})$$



opt

LP/SDP

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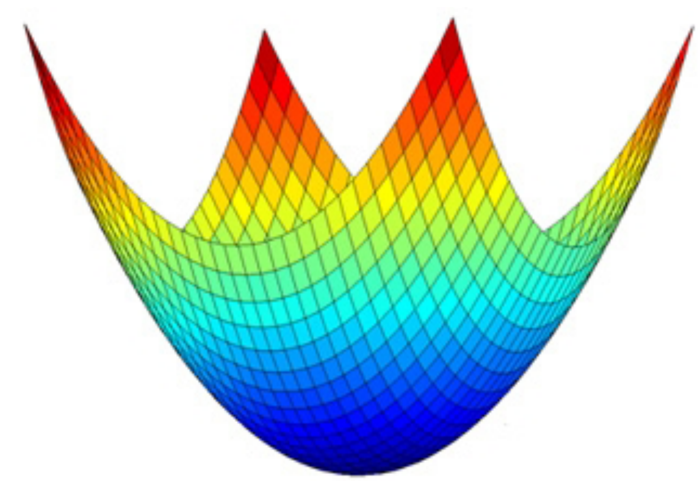

opt'

Constraint Satisfaction Problems

IN: set of variables, set of labels, set of constraints

OUT: assignment that satisfies and optimises constraints

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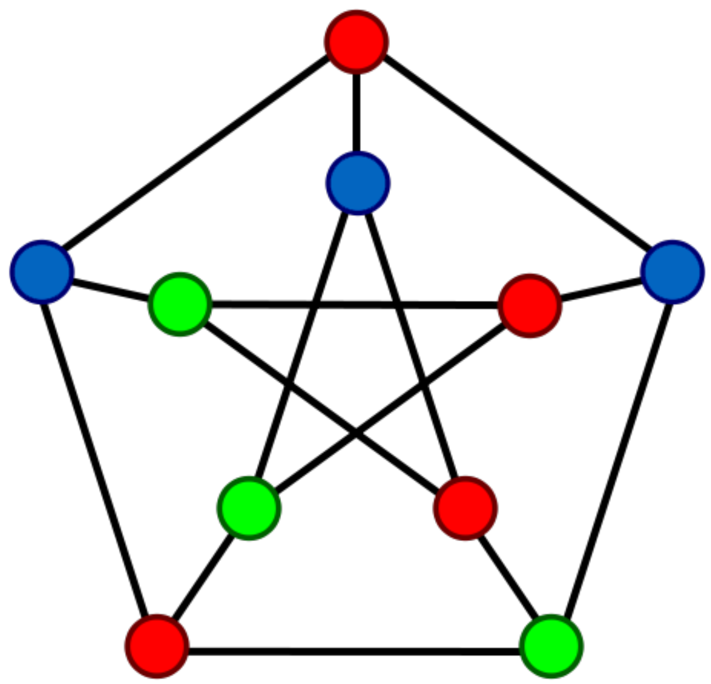
3-Colour



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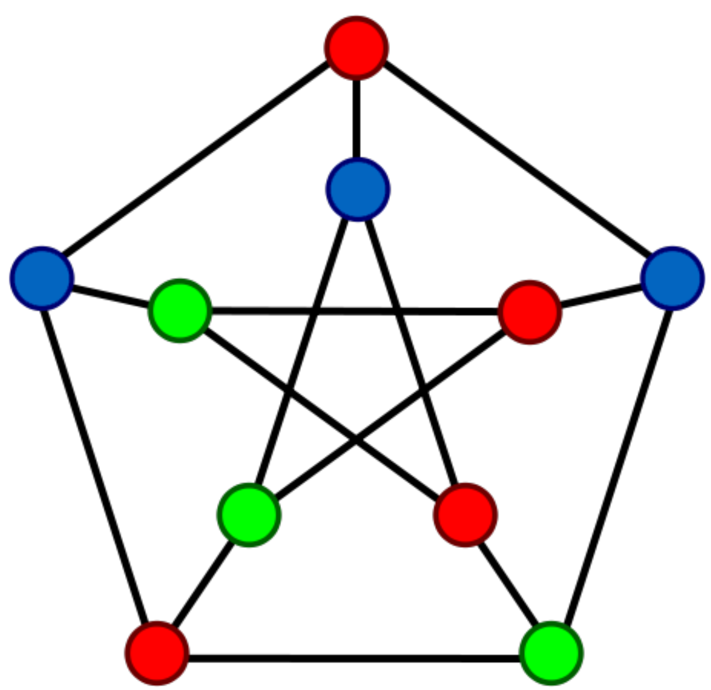

Linear Equations over $\mathbb{Z}_7$

$$x_1 + x_2 + x_3 = 1$$
$$x_2 - x_4 + x_5 = 0$$
$$x_1 - x_3 - x_4 = 2$$

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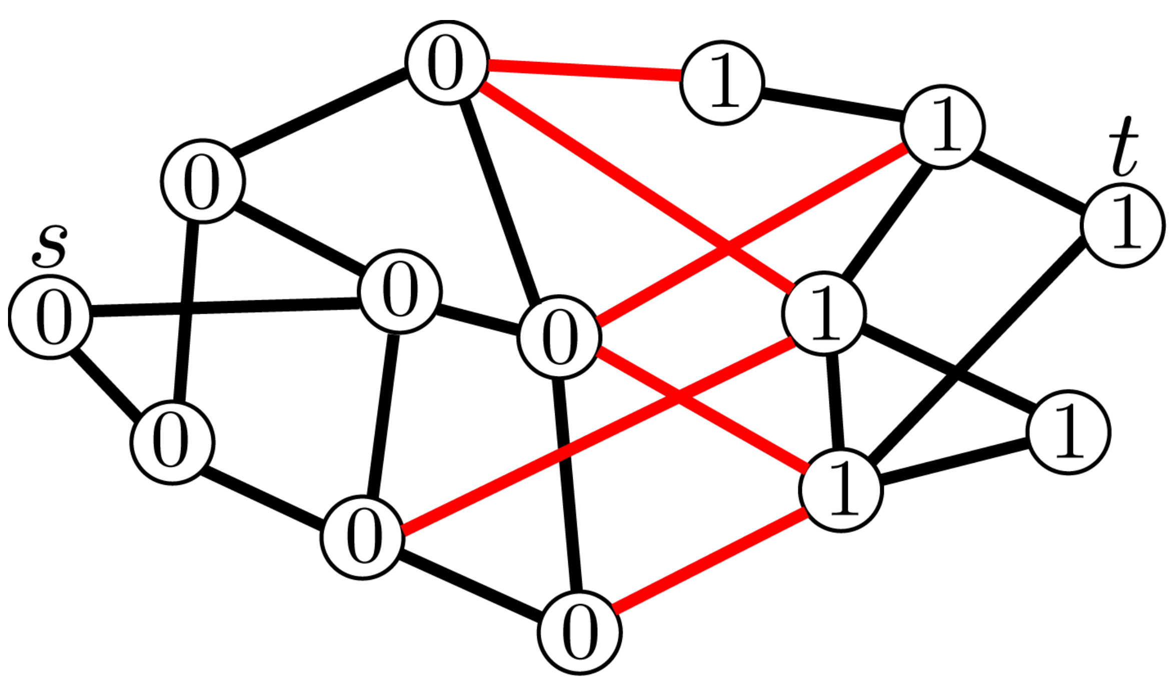

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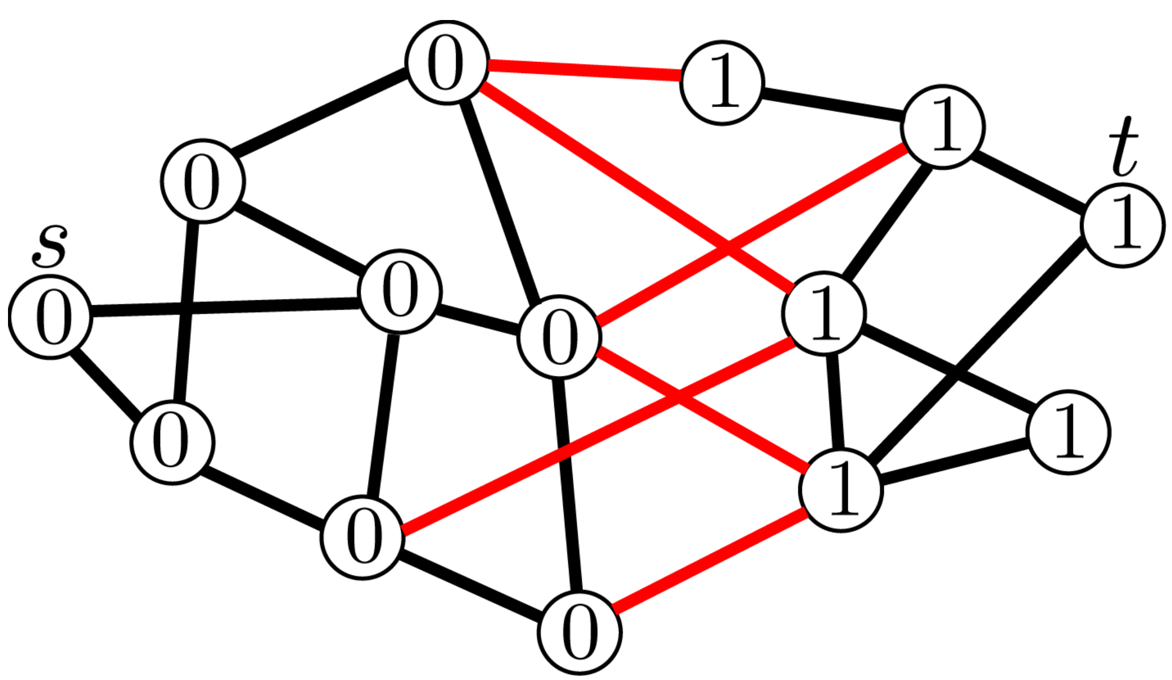
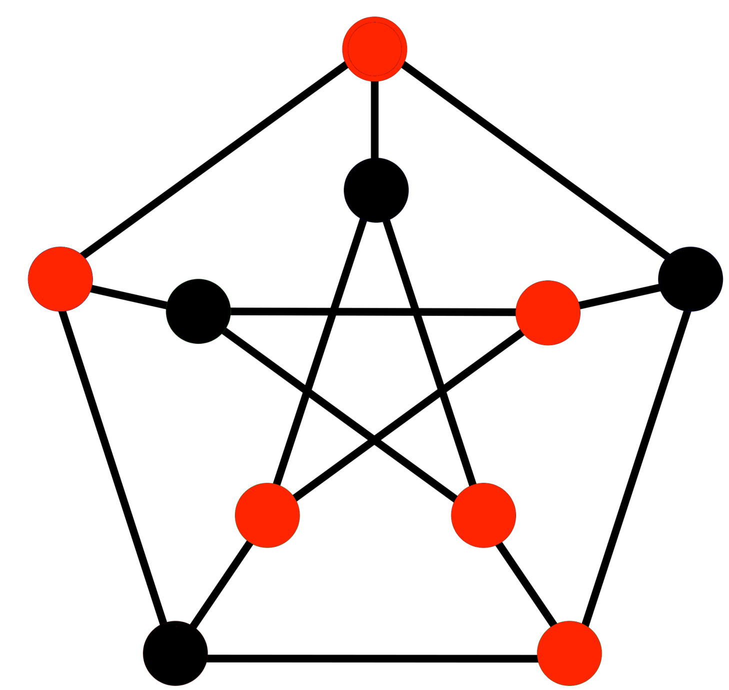


Vertex Cover



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IN: set of variables, set of labels, set of constraints

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- CSP = (V,D,C)
- CSP = homomorphism problem
- CSP = conjunctive query evaluation
- CSP = model checking of pp-fragment of FO

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Vertex Cover

$V = \{x_1, \ldots, x_n\},\ D = \{0, 1\}$

$$I(x_1, \ldots, x_n) = \sum_{\{i,j\} \in E(G)} \psi(x_i, x_j) + \sum_{i \in V(G)} \tau(x_i)$$

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x	y	$\psi(x, y)$
1	1	0
1	0	0
0	1	0
0	0	∞

x	$\tau(x)$
1	1
0	0

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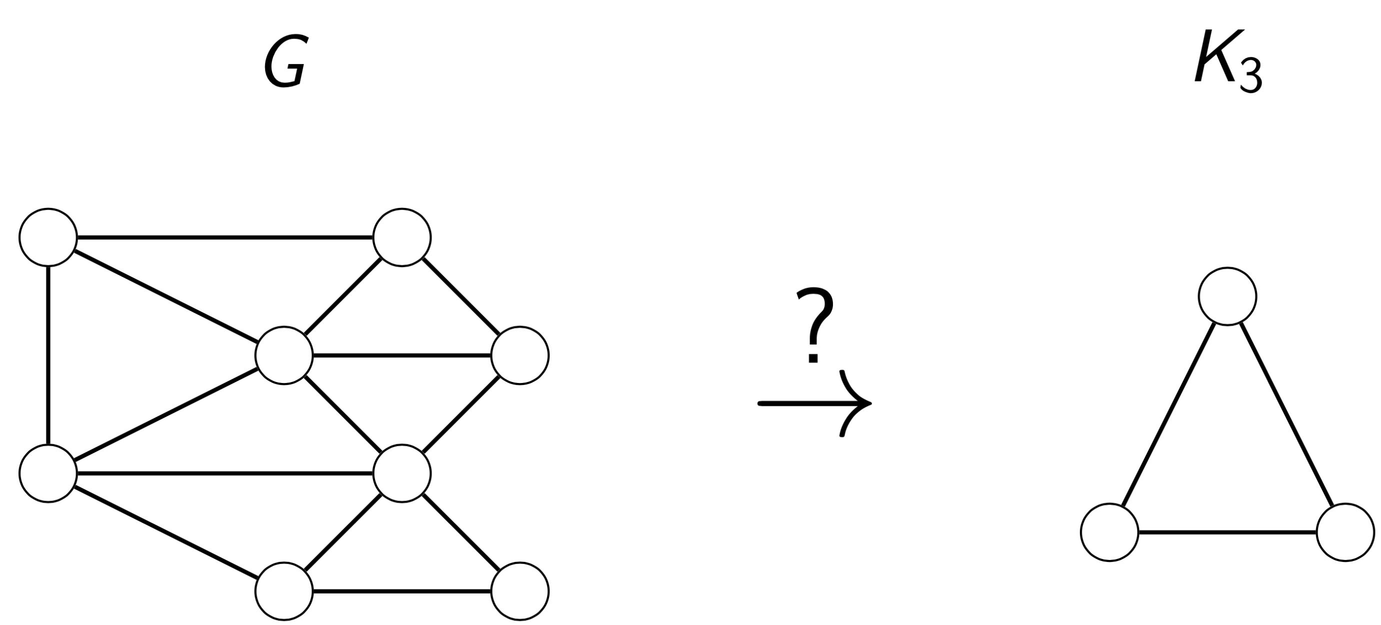

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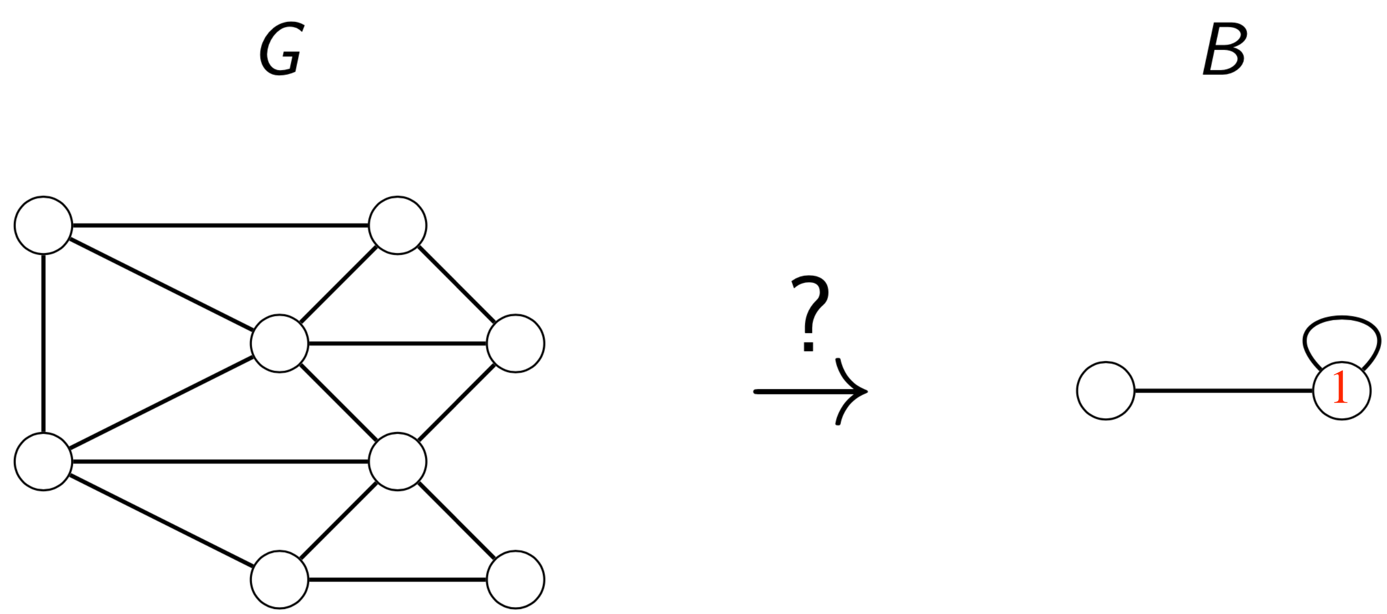

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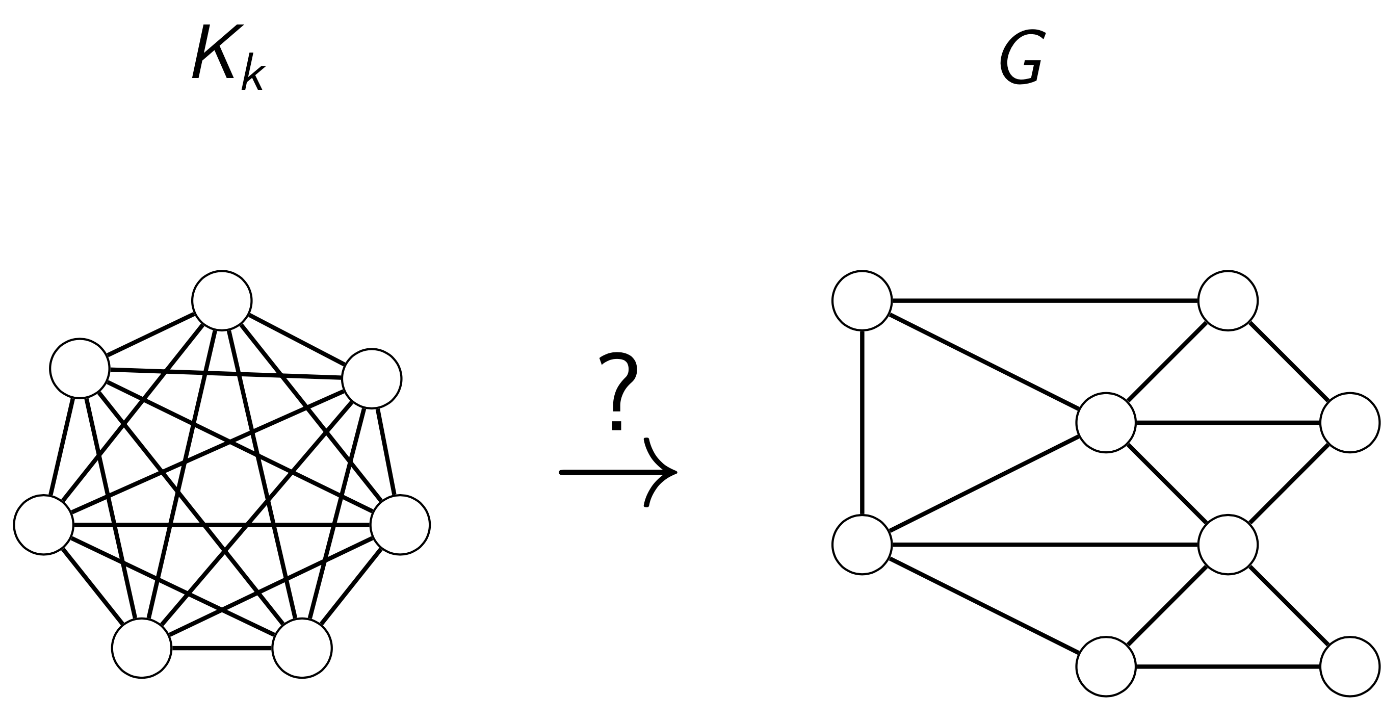
k-Clique



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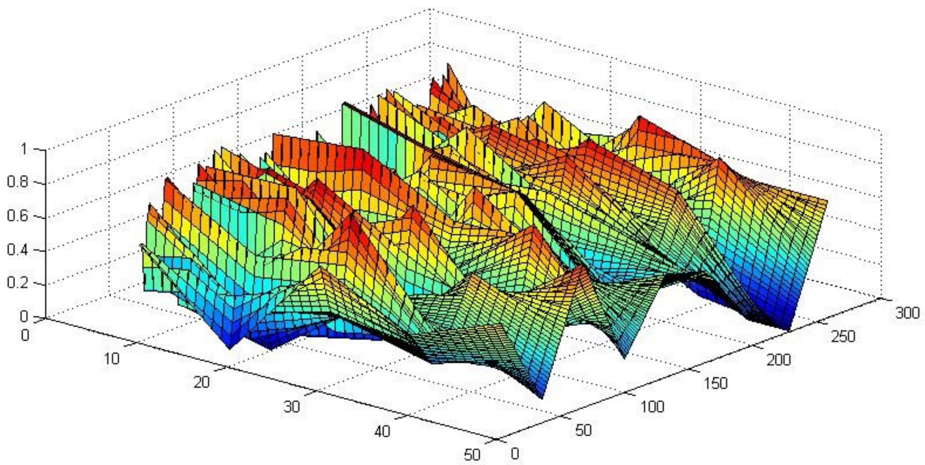
balance between generality and structure

Convex Relaxations

CSP

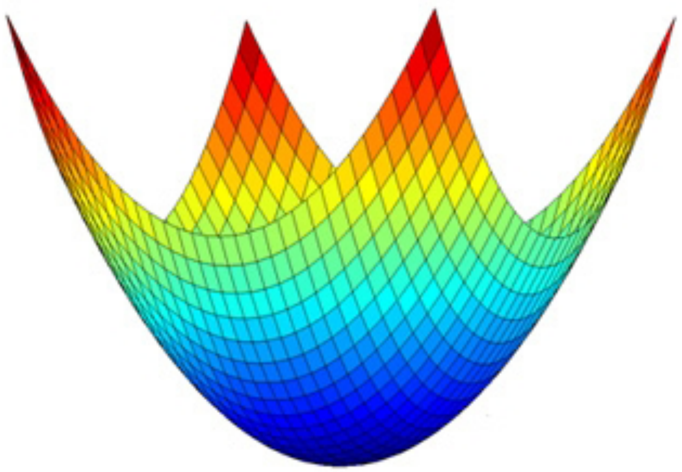
$$\min_{\mathbf{x}\in\Omega} f(\mathbf{x})$$



opt

LP/SDP

$$\min_{\mathbf{x}\in\Omega'} f'(\mathbf{x})$$



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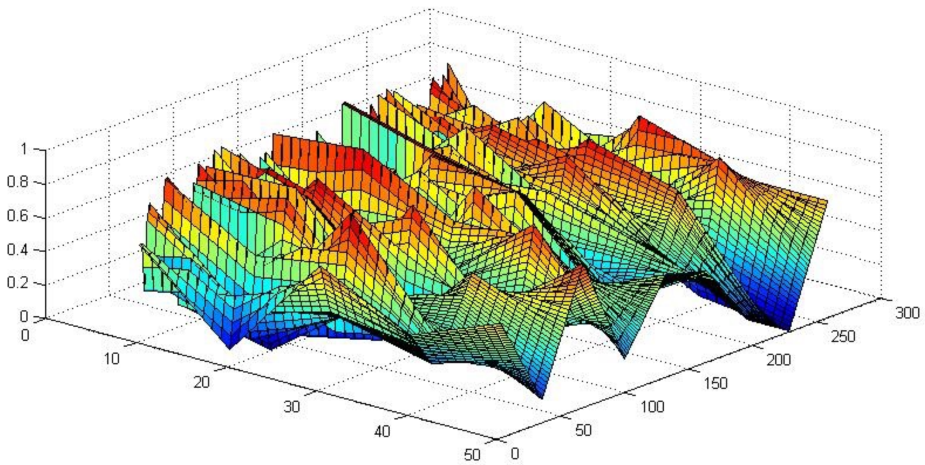
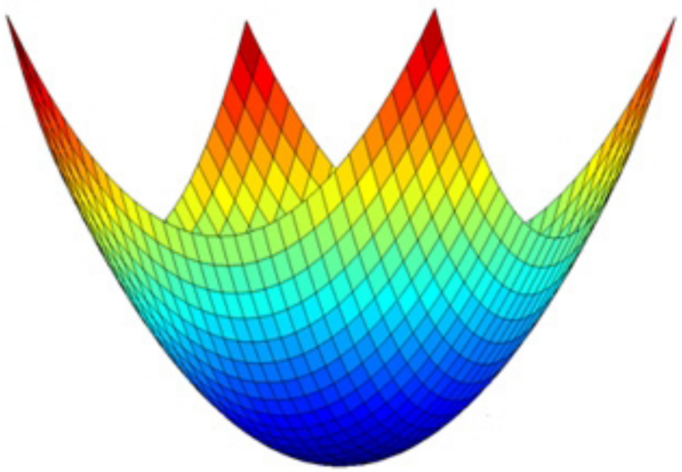
$$\min_{\mathbf{x}\in\Omega'} f'(\mathbf{x})$$



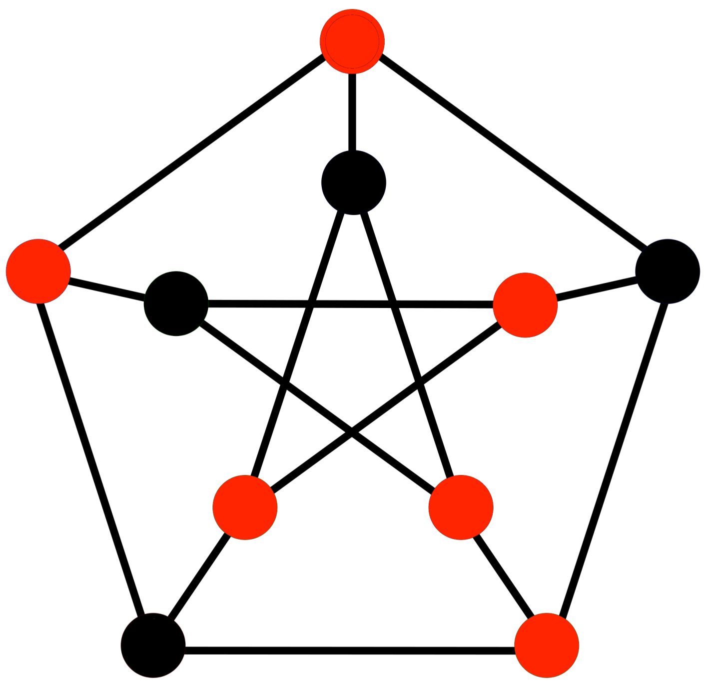


opt

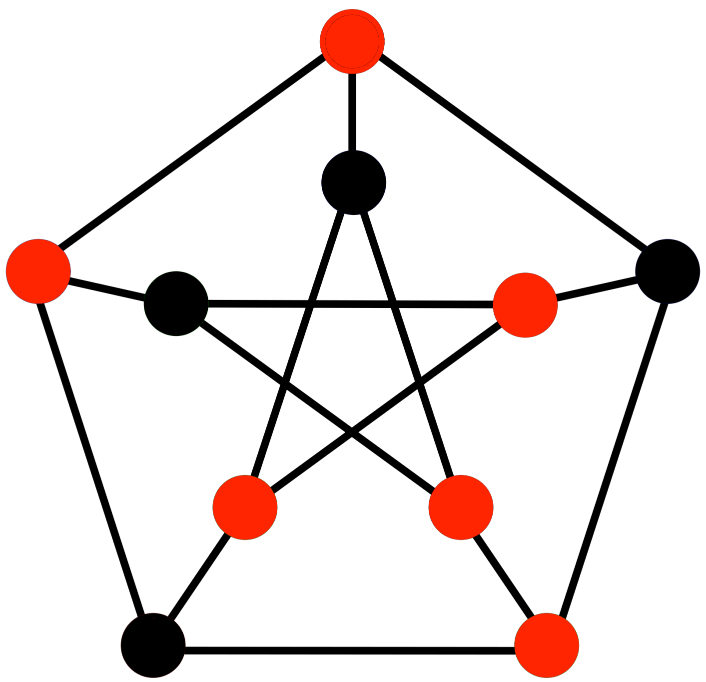
opt'

$opt = opt'$

Vertex Cover

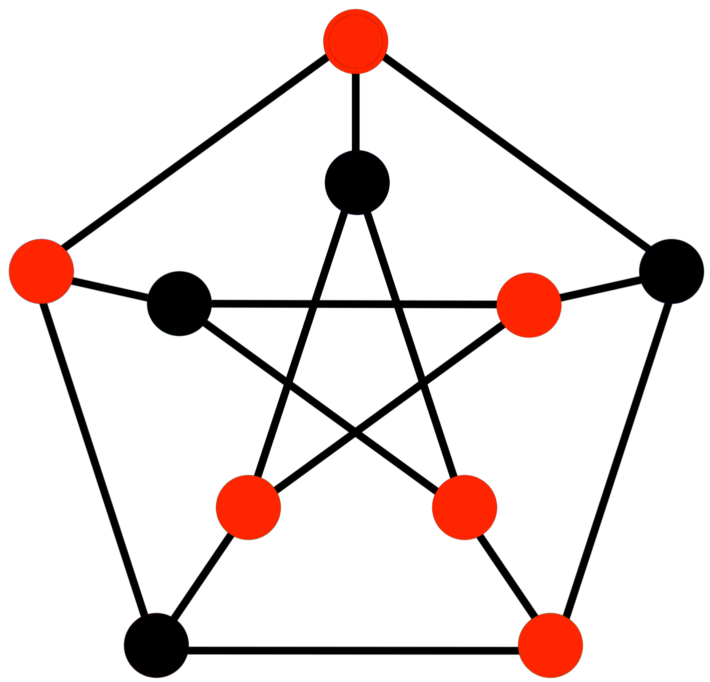


Vertex Cover



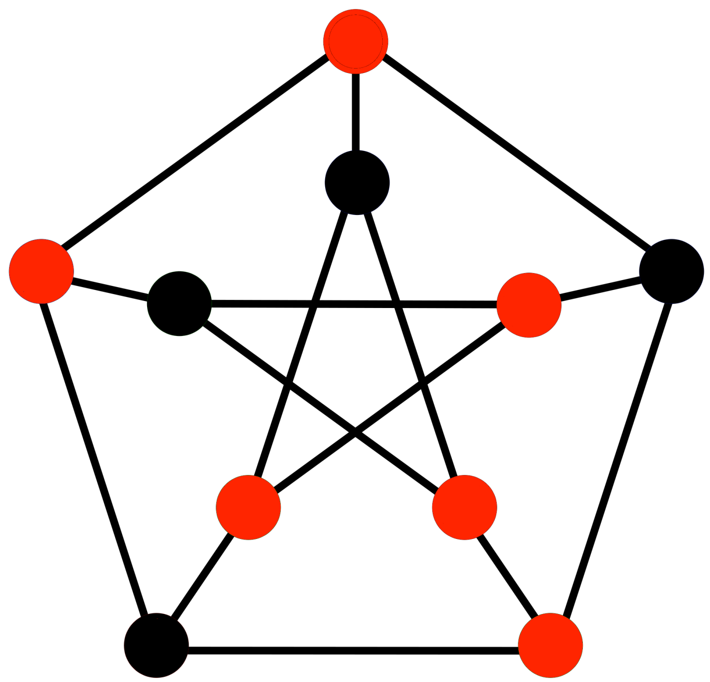
$$\begin{aligned} \min \quad & \sum_{v \in V} x_v \\ \text{s.t.} \quad & x_u + x_v \geq 1 \quad \forall (u,v) \in E \\ & x_v \in \{0,1\} \qquad \forall v \in V \end{aligned}$$

Vertex Cover



$$\begin{aligned} \min \quad & \sum_{v \in V} x_v \\ \text{s.t.} \quad & x_u + x_v \geq 1 \quad \forall (u,v) \in E \\ & 0 \leq x_v \leq 1 \quad \forall v \in V \end{aligned}$$

Vertex Cover



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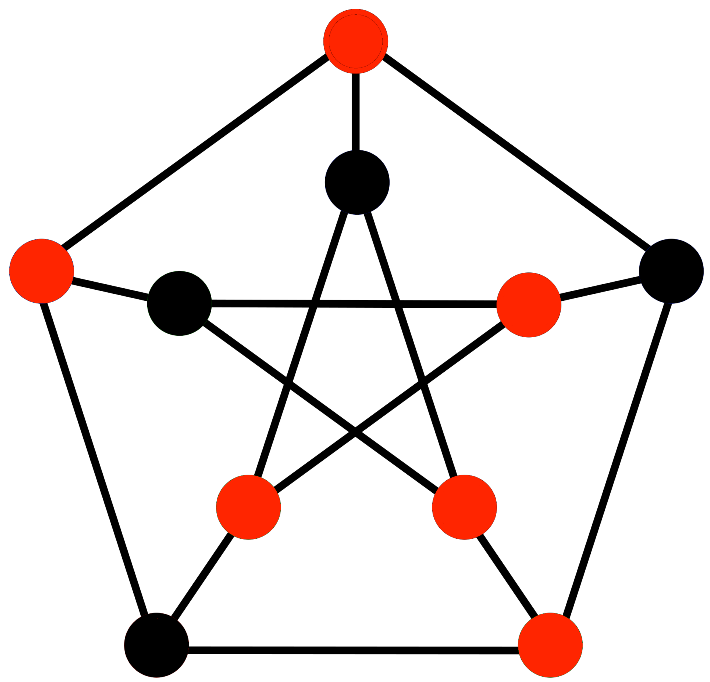
Thm: Basic LP solves CSPs with $(\mathbb{Q} \cup \{\infty\})$-valued functions iff …

[Kolmogorov, Thapper, Ž. SICOMP'15]

Vertex Cover



$$I(x_1,\dots,x_n) = \sum_{\{i,j\}\in E(G)} \psi(x_i,x_j) + \sum_{i\in V(G)} \tau(x_i)$$

x	y	$\psi(x,y)$
1	1	0
1	0	0
0	1	0
0	0	∞

x	$\tau(x)$
1	1
0	0

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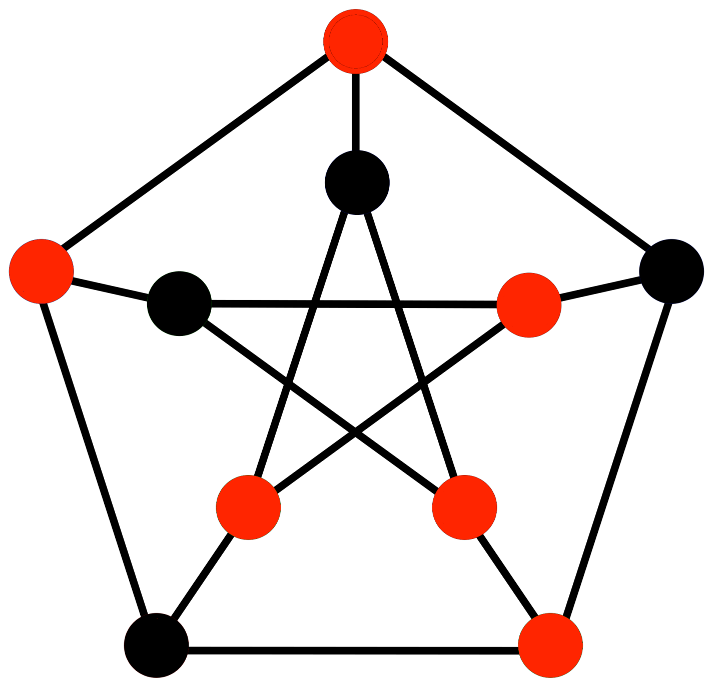
Vertex Cover



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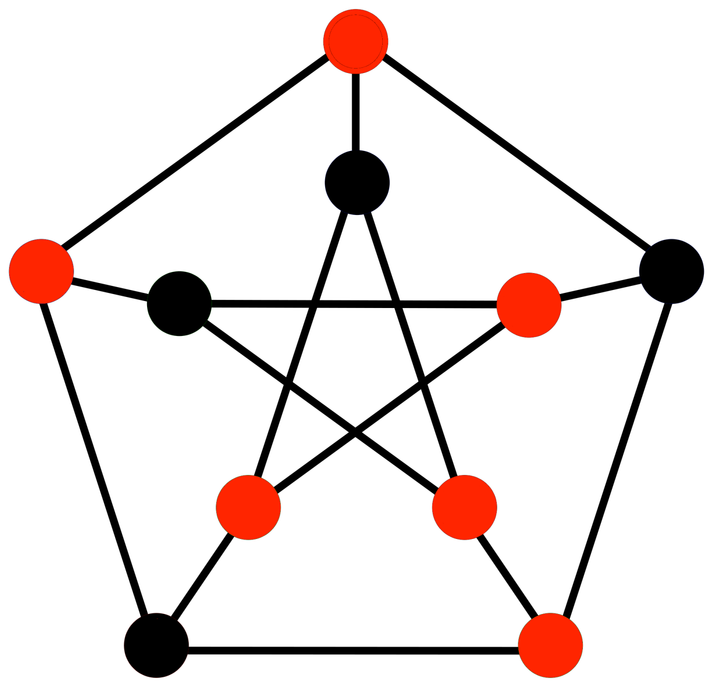
Vertex Cover



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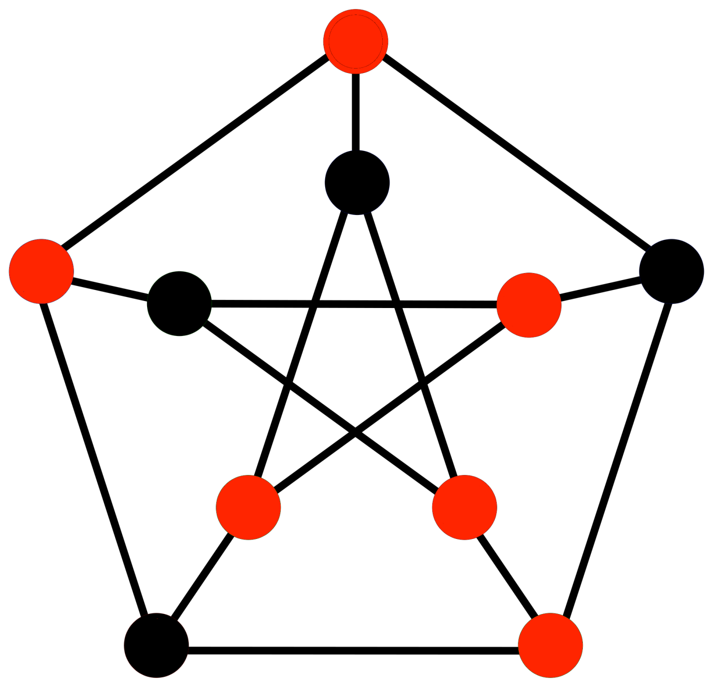
[Kolmogorov, Thapper, Ž. SICOMP'15]

Vertex Cover is not solved by Basic LP!

Vertex Cover



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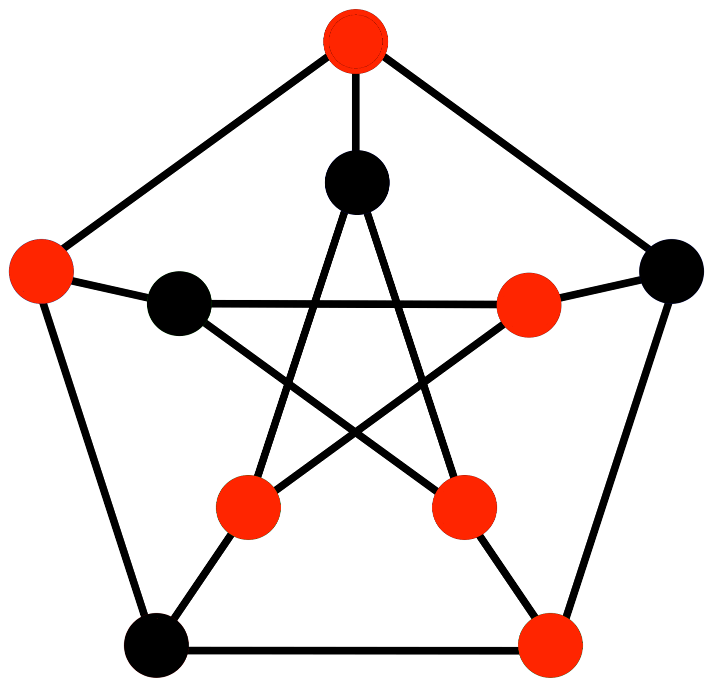
[Kolmogorov, Thapper, Ž. SICOMP'15]

Vertex Cover is not solved by Basic LP!

Submodular CSPs solved by Basic LP!

Vertex Cover



$$I(x_1, \ldots, x_n) = \sum_{\{i,j\} \in E(G)} \psi(x_i, x_j) + \sum_{i \in V(G)} \tau(x_i)$$

x	y	$\psi(x,y)$
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1	0	0
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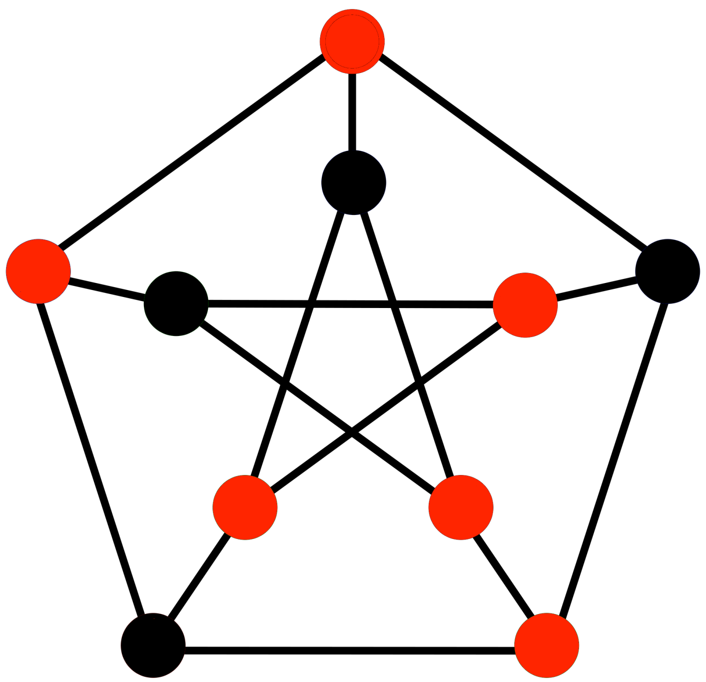
[Kolmogorov, Thapper, Ž. SICOMP'15]

functions on non-distributive lattices

Vertex Cover is not solved by Basic LP!

Submodular CSPs solved by Basic LP!

Vertex Cover



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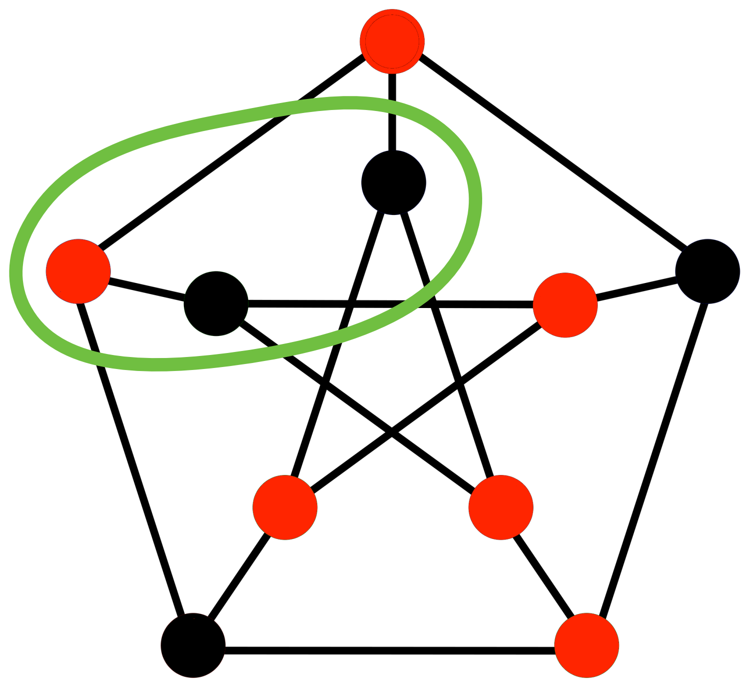
Vertex Cover



$$
\begin{aligned}
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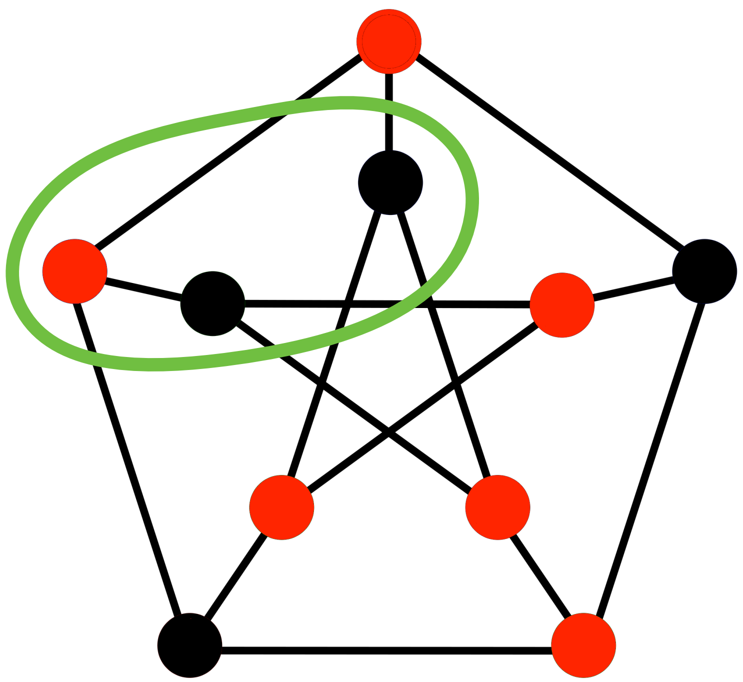
& 0 \leq x_v \leq 1 \quad \forall v \in V
\end{aligned}
$$

Vertex Cover



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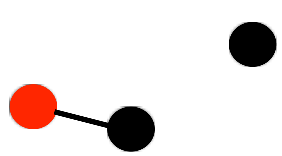
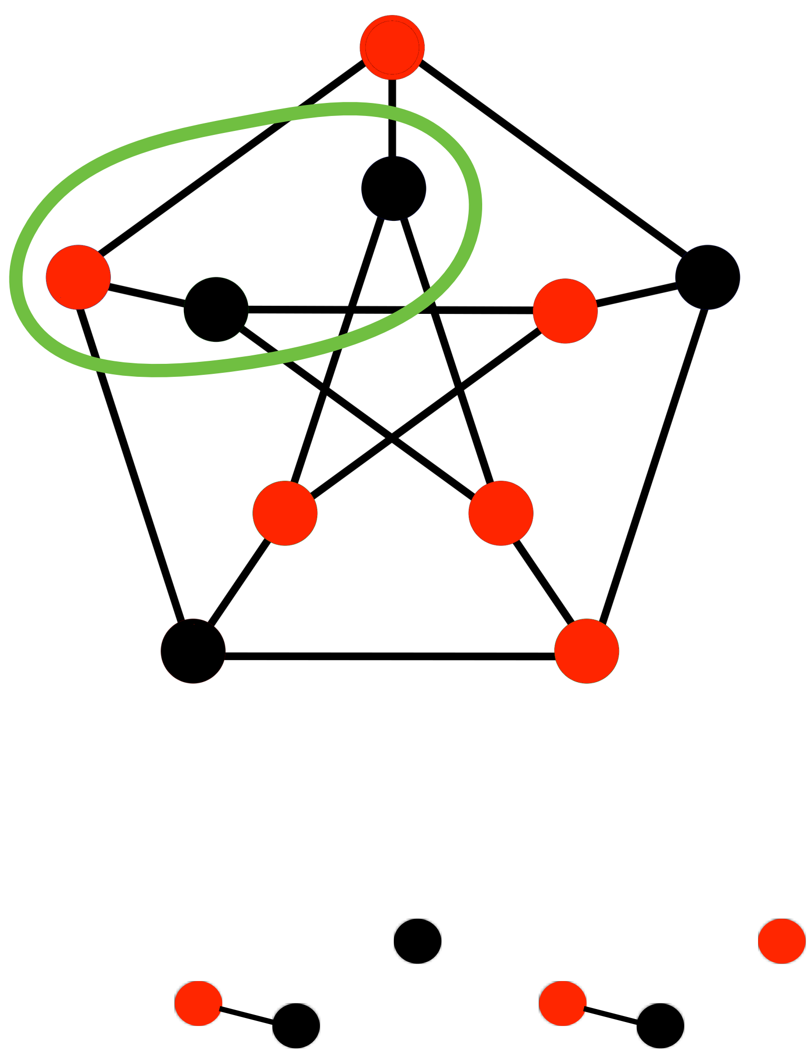
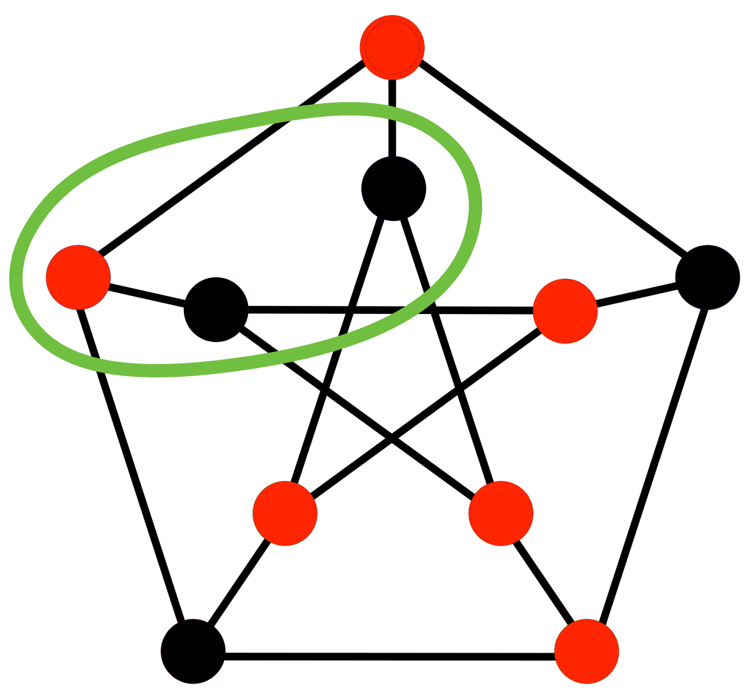


Vertex Cover



$$\begin{aligned} \min \quad & \sum_{v \in V} x_v \\ \text{s.t.} \quad & x_u + x_v \geq 1 \quad \forall (u,v) \in E \\ & 0 \leq x_v \leq 1 \quad \forall v \in V \end{aligned}$$

Vertex Cover



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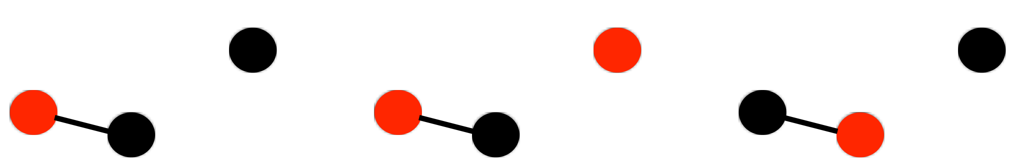
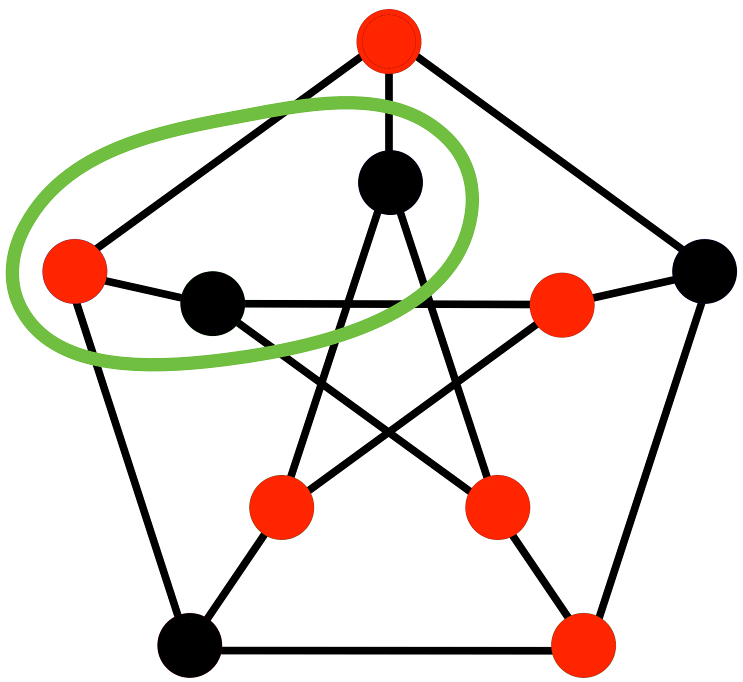


Vertex Cover



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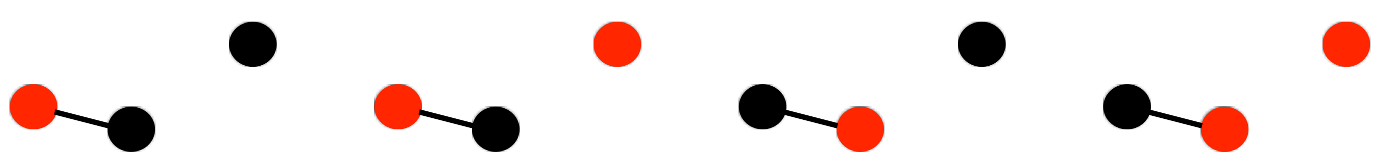
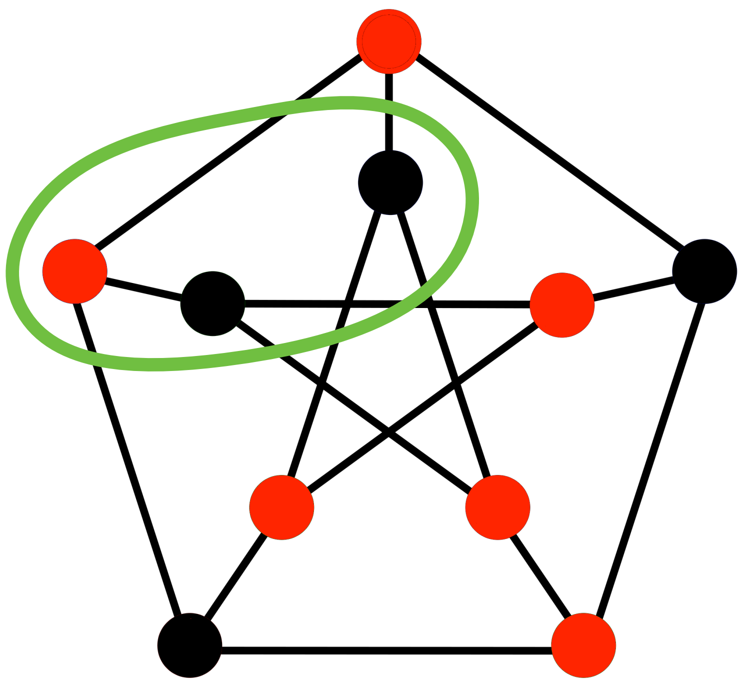


Vertex Cover



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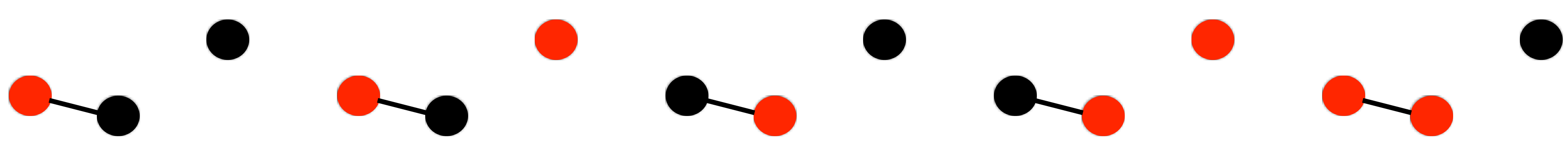
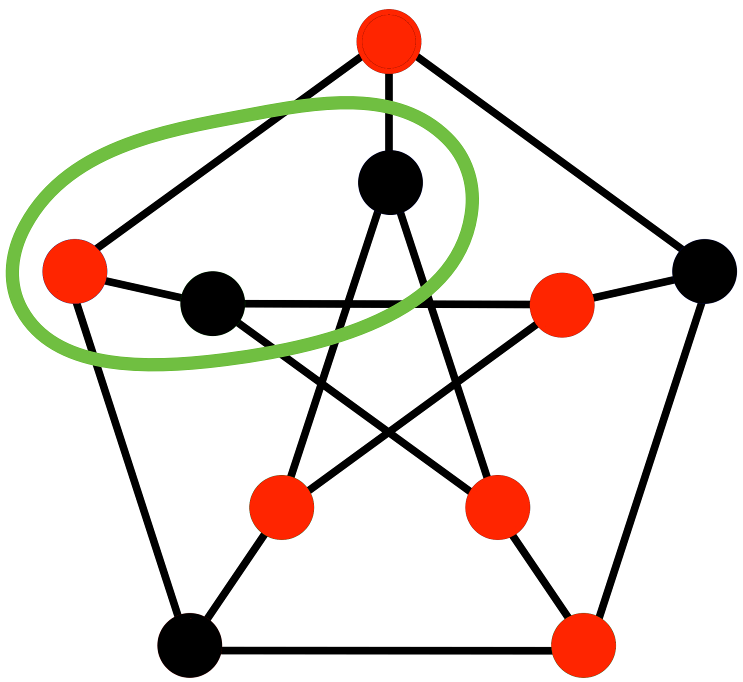


Vertex Cover



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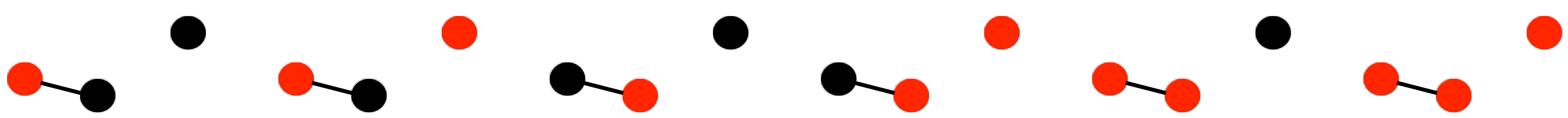
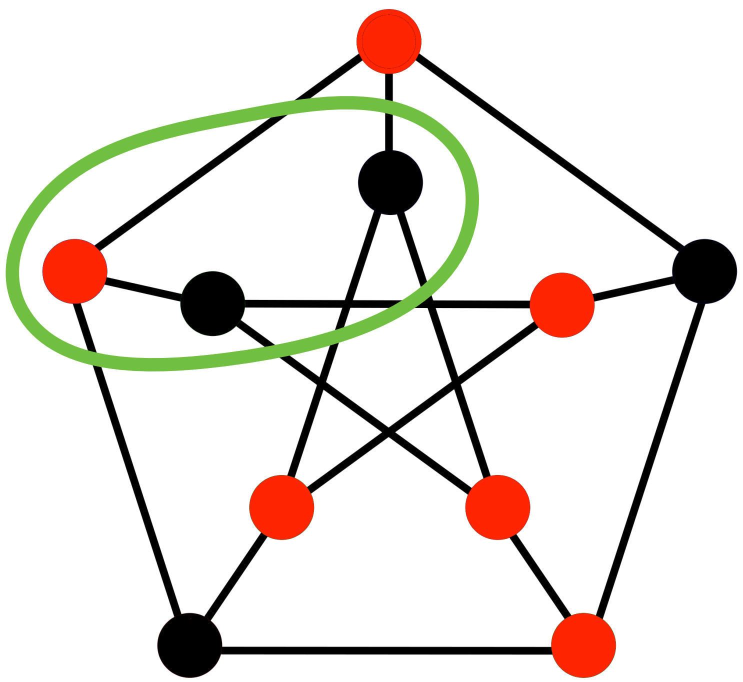
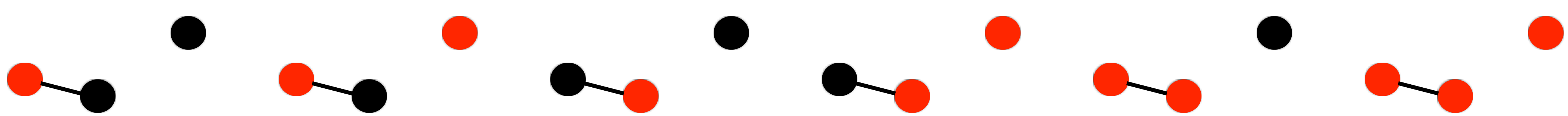


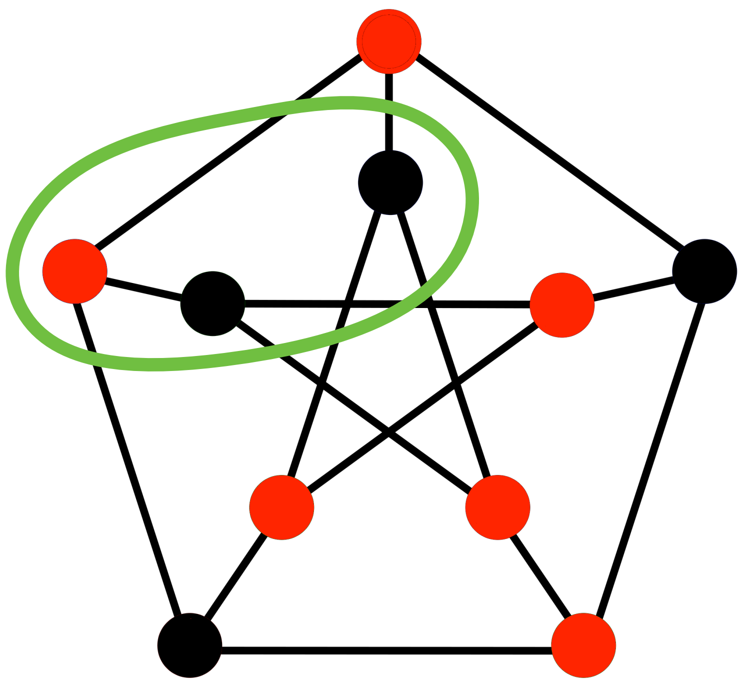
Vertex Cover



$$\begin{array}{lll} \min & \sum_{v \in V} \mathbb{P}_{S\sim\mathcal{D}(\{v\})}[v \in S] & \\ \text{s.t.} & \mathcal{D}(U) \text{ is a distribution on } \mathcal{P}(U) & \forall U \subseteq V, |U| \le k \\ & \mathbb{P}_{S\sim\mathcal{D}(\{u,v\})}[u \notin S, v \notin S] = 0 & \forall (u,v) \in E \\ & \mathbb{P}_{S\sim\mathcal{D}(U')} = \mathbb{P}_{S\sim\mathcal{D}(U)} & \forall U' \subseteq U \subseteq V, |U| \le k \end{array}$$



Vertex Cover



$$\min \quad \sum_{v\in V} \mathbb{P}_{S\sim\mathcal{D}(\{v\})}[v \in S]$$

$$\text{s.t.} \quad \mathcal{D}(U) \text{ is a distribution on } \mathcal{P}(U) \qquad \forall U \subseteq V, |U| \leq k$$

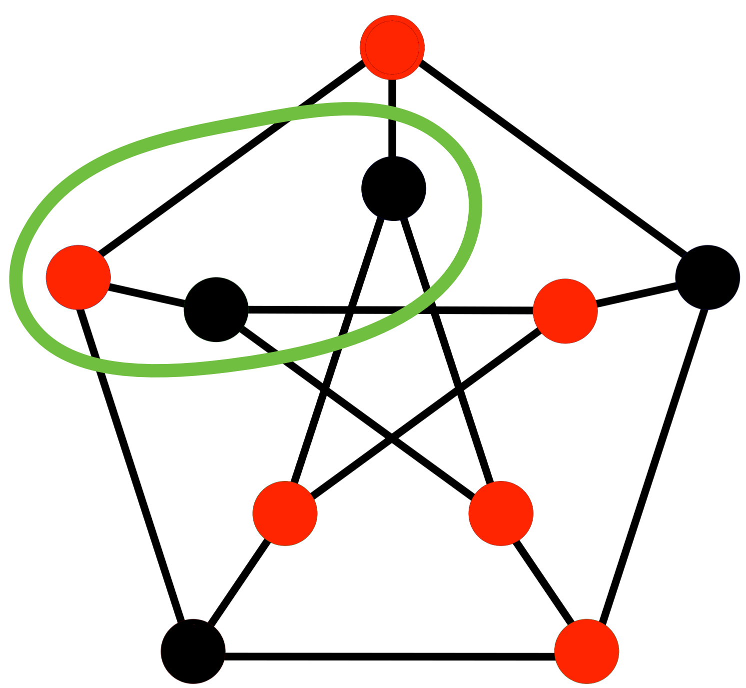
$$\mathbb{P}_{S\sim\mathcal{D}(\{u,v\})}[u \notin S, v \notin S] = 0 \qquad \forall (u,v) \in E$$

$$\mathbb{P}_{S\sim\mathcal{D}(U')} = \mathbb{P}_{S\sim\mathcal{D}(U)} \qquad \forall U' \subseteq U \subseteq V, |U| \leq k$$

Thm: Sherali-Adams(k) LP solves CSPs with ($\mathbb{Q}\cup\{\infty\}$)-valued functions iff …

[Thapper, Ž. SICOMP'17]

Vertex Cover



$$
\begin{aligned}
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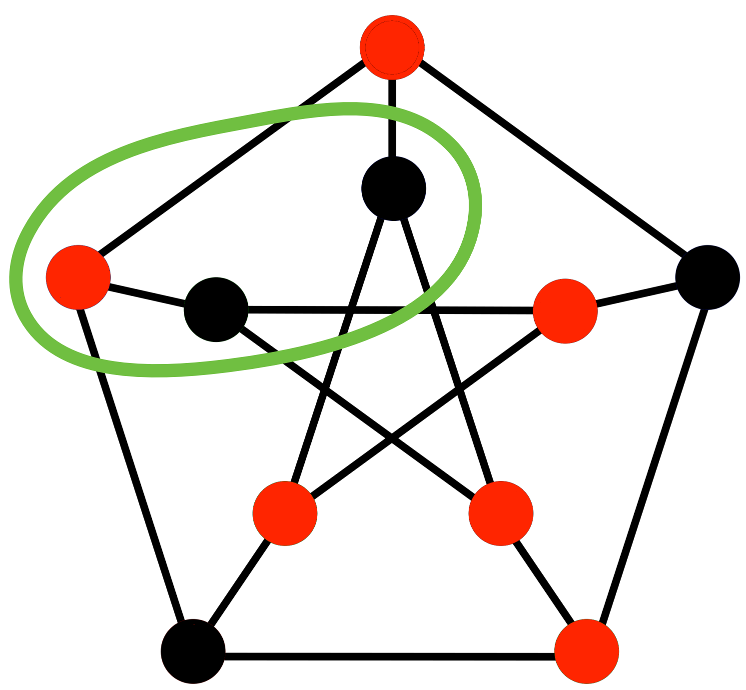
(iff Sherali-Adams(3) solves)

[Thapper, Ž. SICOMP'17]

Vertex Cover



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\begin{array}{lll}
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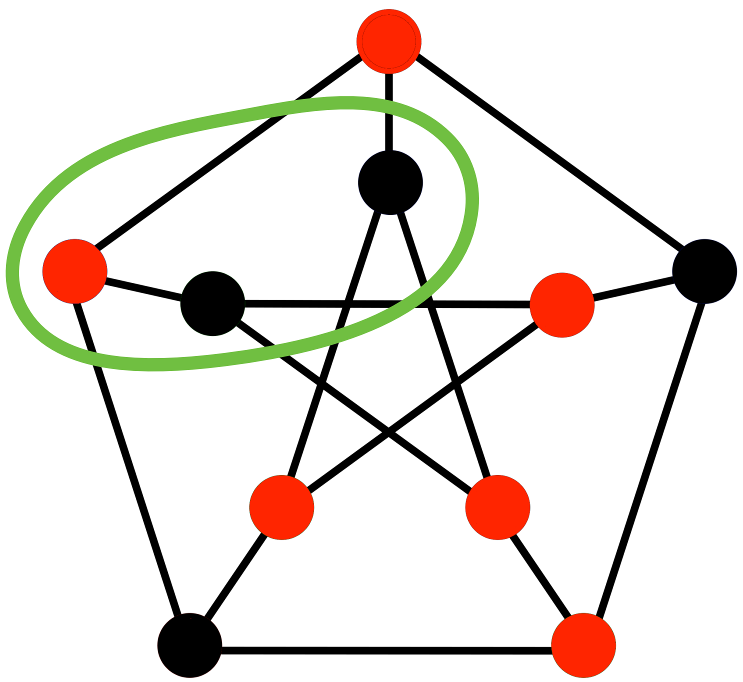
(iff Sherali-Adams(3) solves)

[Thapper, Ž. SICOMP'17]

Vertex Cover is not solved by Sherali-Adams(k) LP!

Vertex Cover



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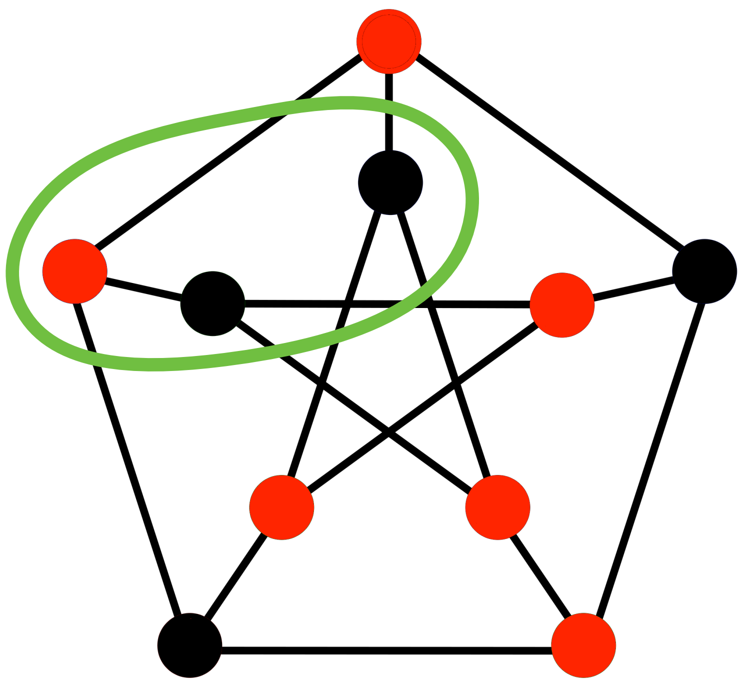
Vertex Cover



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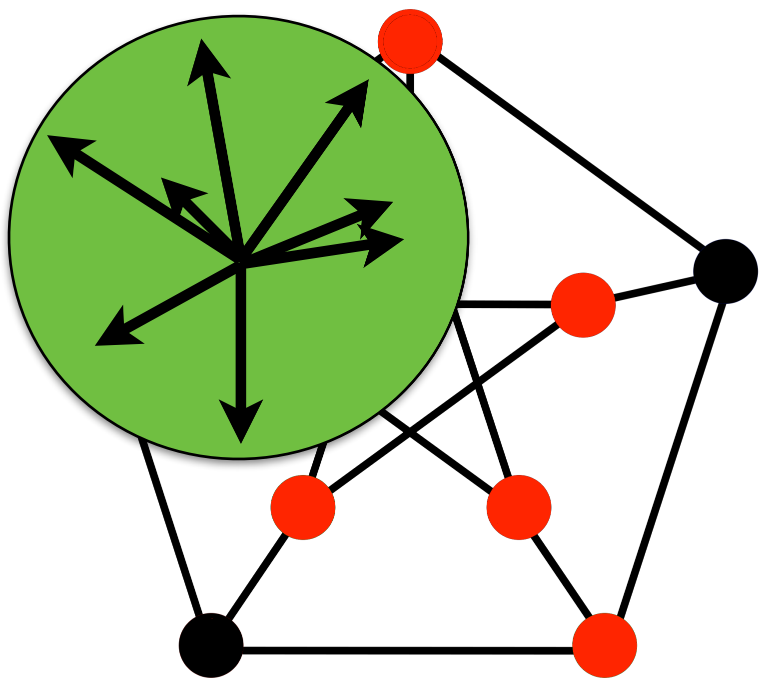
$\mathcal{D} \in \mathbb{R}$ ➡ $\mathcal{D} \in \mathbb{R}^d$

Vertex Cover



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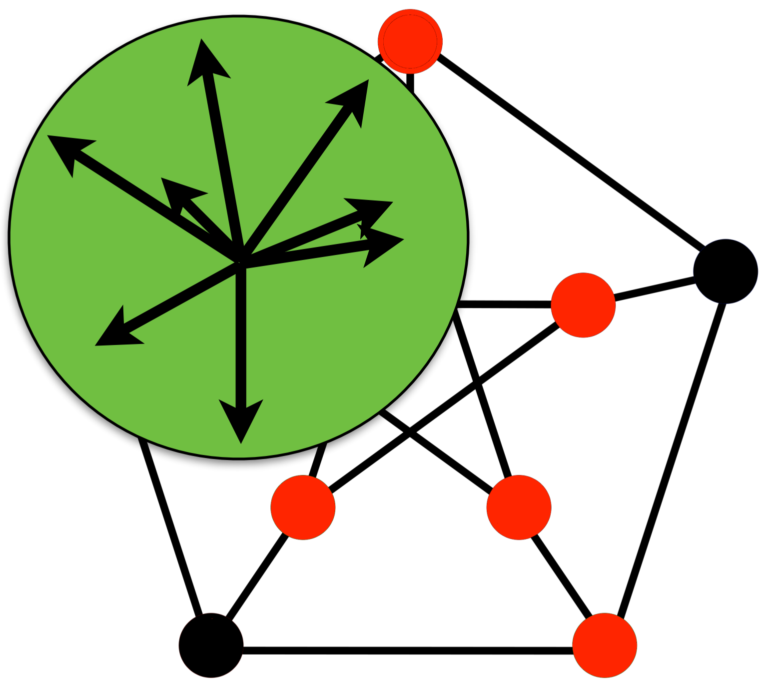
$\mathcal{D} \in \mathbb{R}$ ➡ $\mathcal{D} \in \mathbb{R}^d$

Vertex Cover



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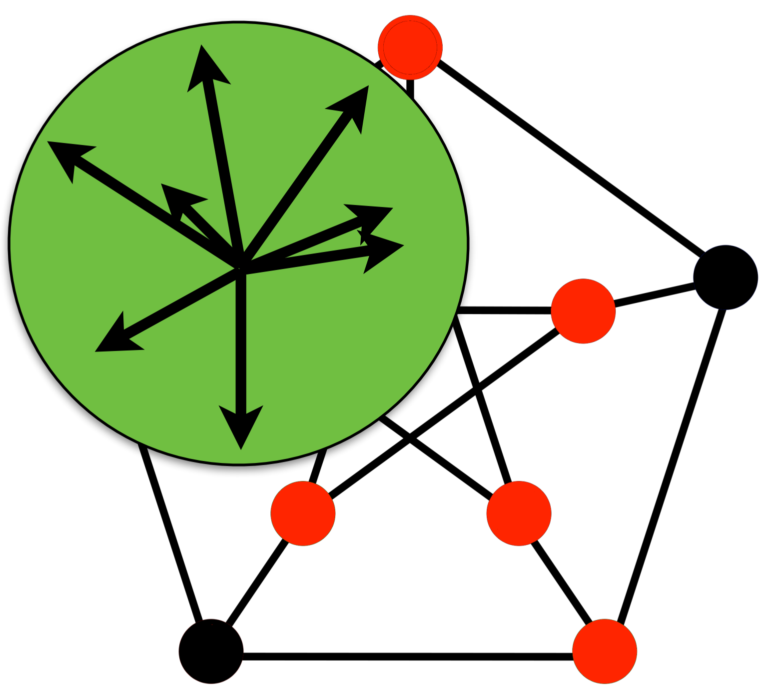
Thm: Sum-of-Squares SDP solves CSPs with ($\mathbb{Q}\cup\{\infty\}$)-valued functions iff …

*[Thapper, **Ž**. ACM ToCT'18]*

Vertex Cover



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\begin{array}{lll}
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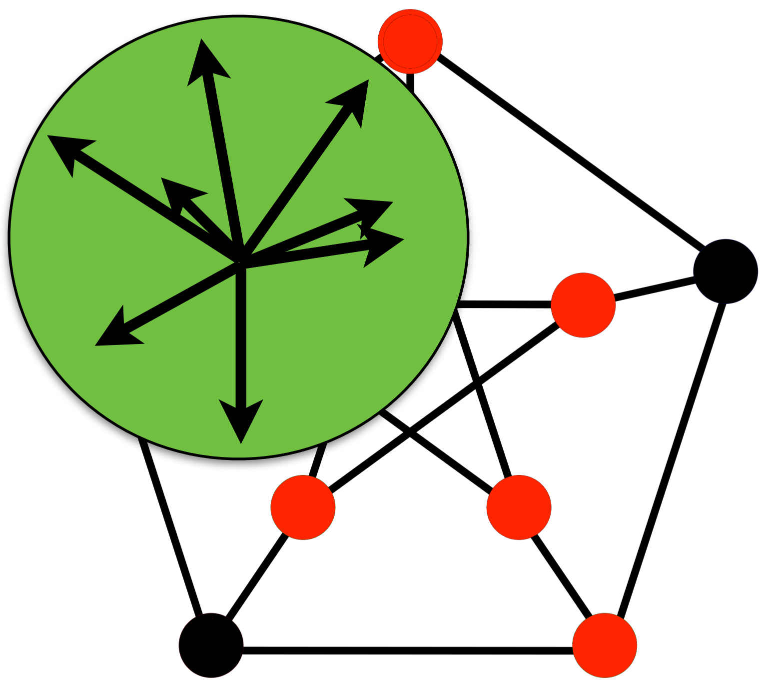
$$

Thm: Sum-of-Squares SDP solves CSPs with ($\mathbb{Q} \cup \{\infty\}$)-valued functions iff …

(Sherali-Adams(3) or not even Sum-of-Squares($\frac{n}{1024}$)) *[Thapper, Ž. ACM ToCT'18]*

Vertex Cover



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Thm: Sum-of-Squares SDP solves CSPs with ($\mathbb{Q} \cup \{\infty\}$)-valued functions iff …

(Sherali-Adams(3) or not even Sum-of-Squares($\frac{n}{1024}$)) *[Thapper, Ž. ACM ToCT'18]*

Vertex Cover is not solved by any poly-size SDP relaxation!

[Lee, Raghavendra, Steurer STOC'15]

Exact Solvability

Power of Basic LP *[Kolmogorov, Thapper, **Ž**. SICOMP'15]*

Power of Sherali-Adams LP *[Thapper, **Ž**. SICOMP'17]*

Power of Sum-of-Squares SDP *[Thapper, **Ž**. ACM ToCT'18]*

Exact Solvability

Power of Basic LP *[Kolmogorov, Thapper, **Ž**. SICOMP'15]*

Power of Sherali-Adams LP *[Thapper, **Ž**. SICOMP'17]*

Power of Sum-of-Squares SDP *[Thapper, **Ž**. ACM ToCT'18]*

Fixed RHS: CSP(-,B)

Exact Solvability

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Power of Sherali-Adams LP *[Thapper, **Ž**. SICOMP'17]*

all functions

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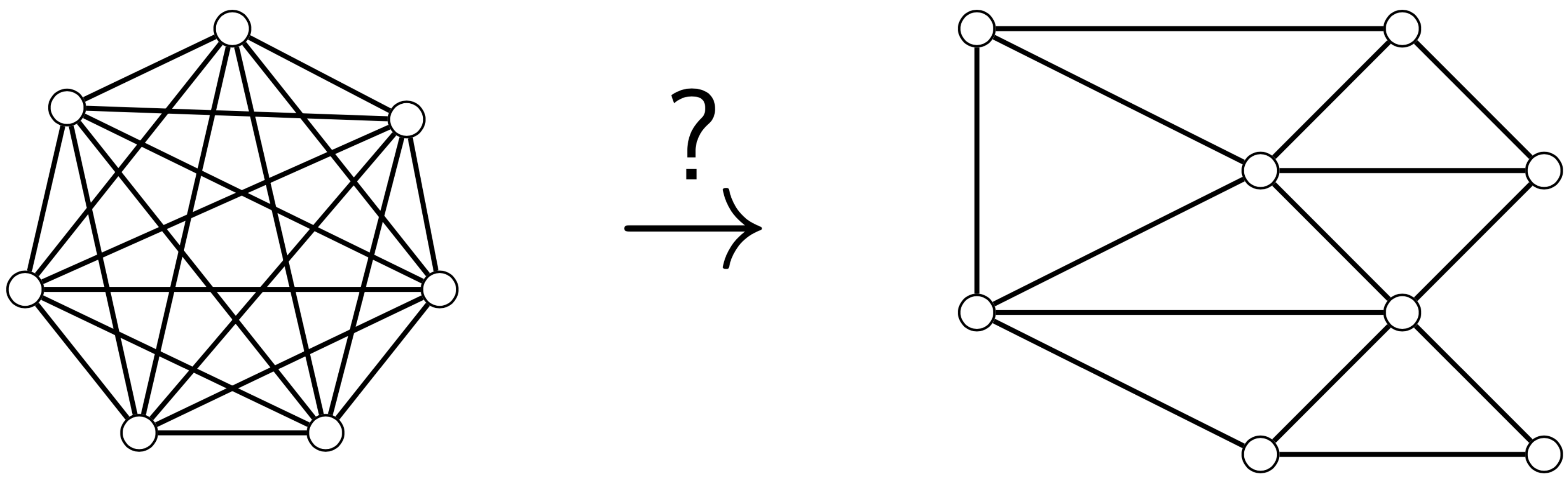
all functions

Power of Sum-of-Squares SDP *[Thapper, Ž. ACM ToCT'18]*

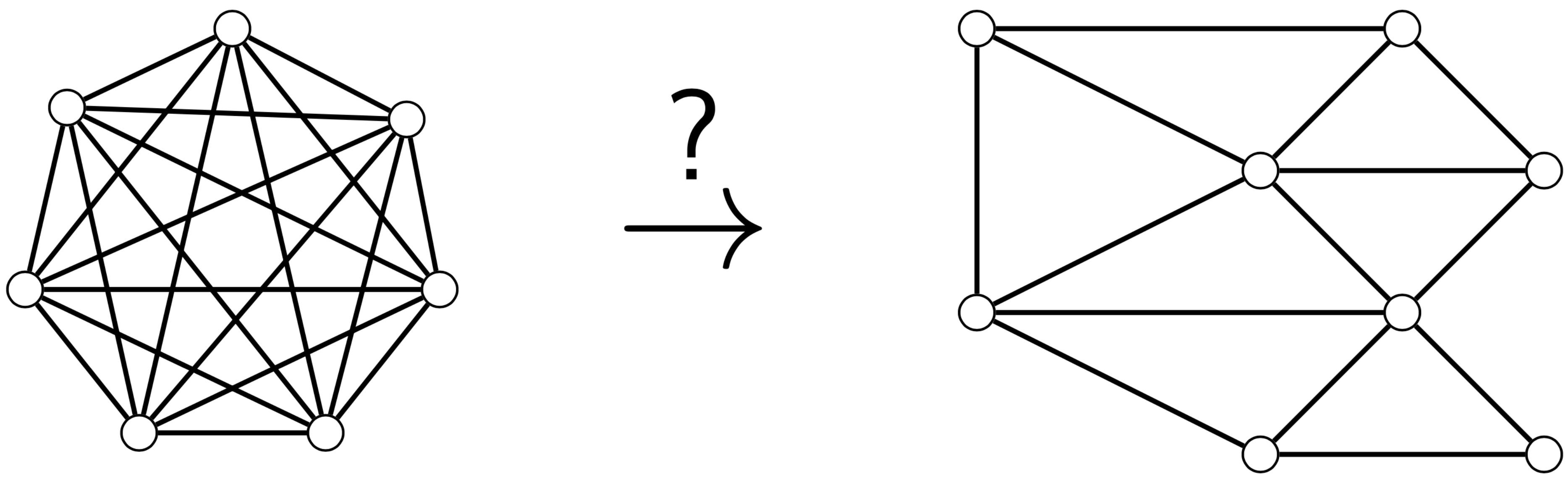
Fixed RHS: CSP(-,B)

Fixed LHS: CSP(A,-)

CSP($\mathcal{A}$,-)

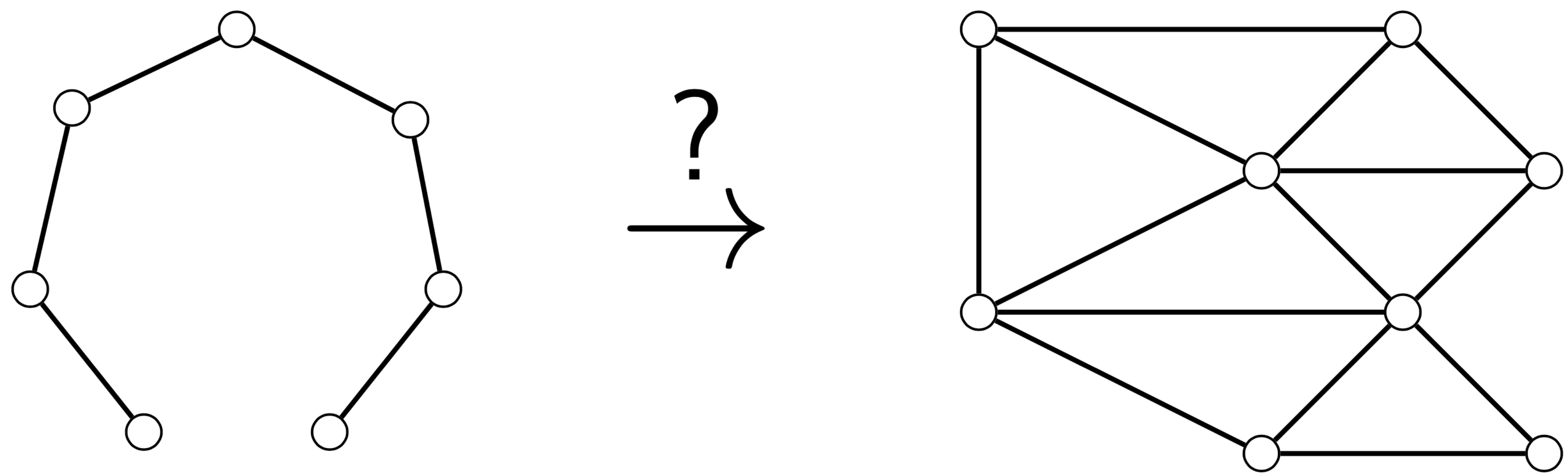


CSP($\mathcal{A}$,-)

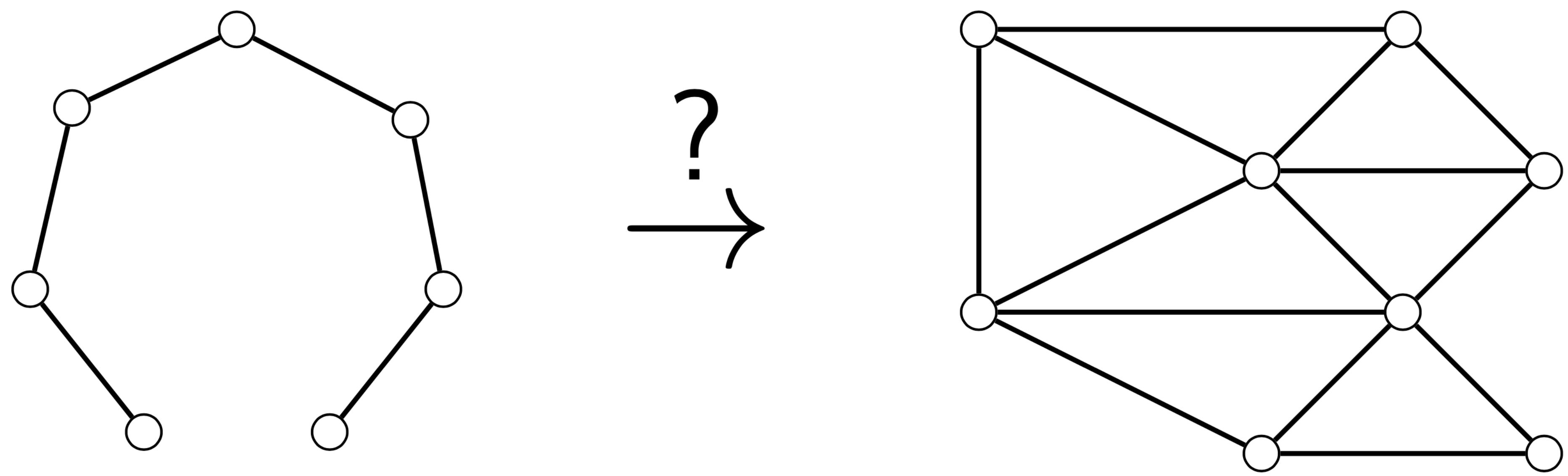


k-Clique = CSP($\mathcal{A}$,-) for $\mathcal{A} = \{K_k\}_{k \geqq 1}$

CSP($\mathcal{A}$,-)



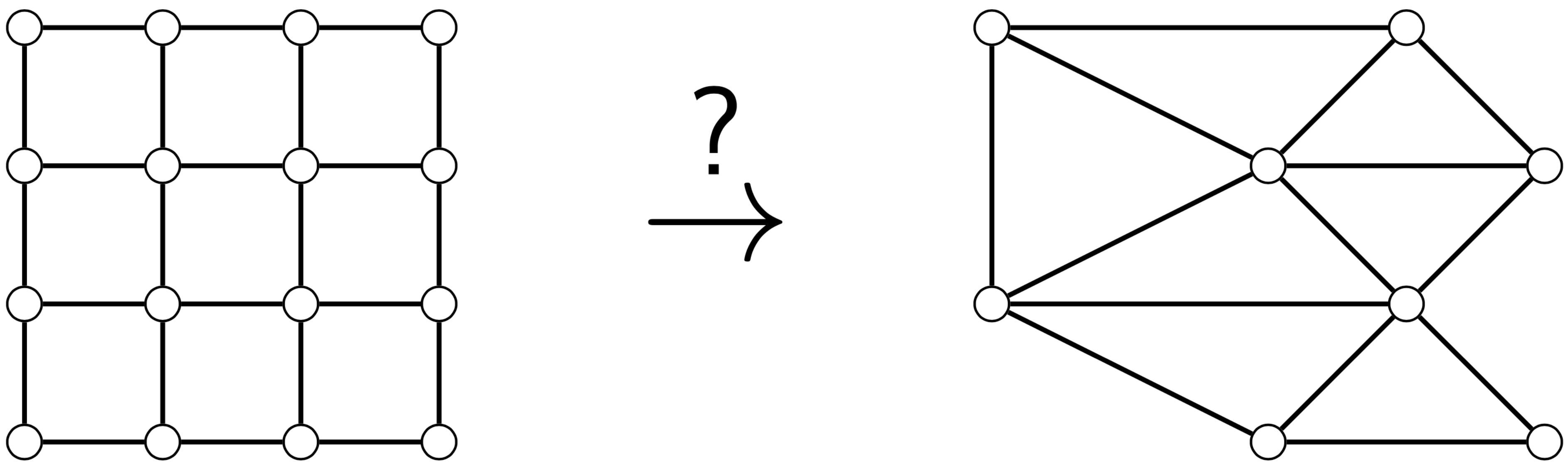
CSP($\mathcal{A}$,-)



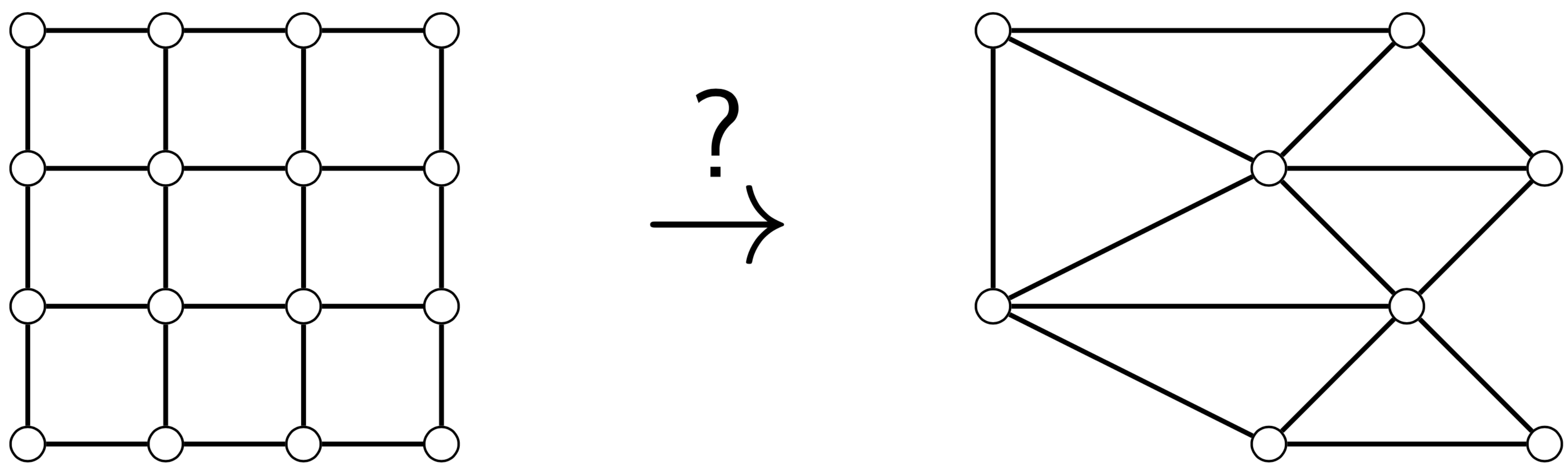
CSP($\mathcal{A}$,-) in PTIME if tw($\mathcal{A}$) $< \infty$

[Freuder JACM'82, AAAI'90]

CSP($\mathcal{A}$,-)

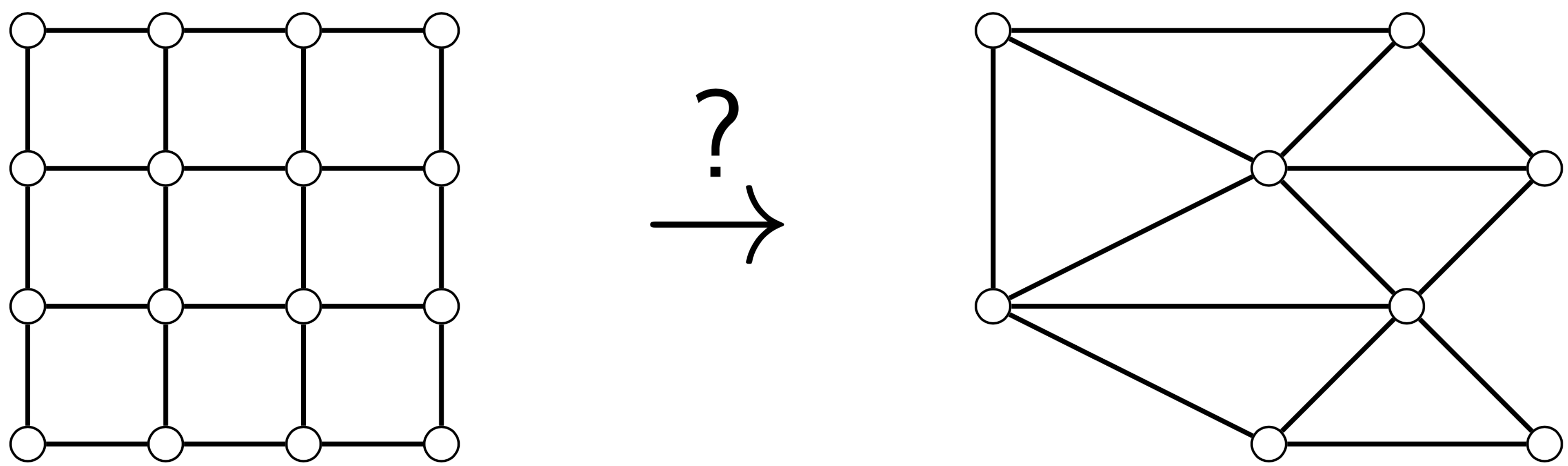


CSP($\mathcal{A}$,-)



CSP($\mathcal{A}$,-) in PTIME if tw(core($\mathcal{A}$)) < ∞ *[Dalmau, Kolaitis, Vardi CP'02]*

CSP($\mathcal{A}$,-)

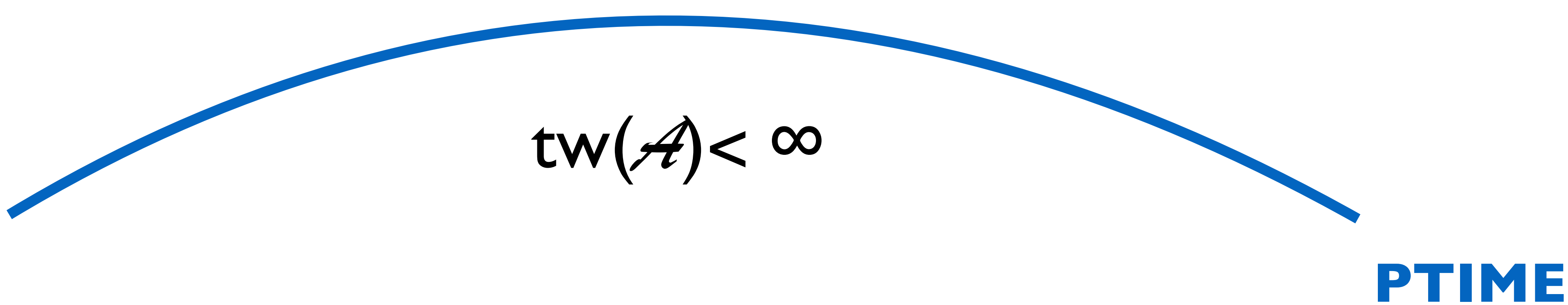


CSP($\mathcal{A}$,-) in PTIME if tw(core($\mathcal{A}$)) $< \infty$ *[Dalmau, Kolaitis, Vardi CP'02]*

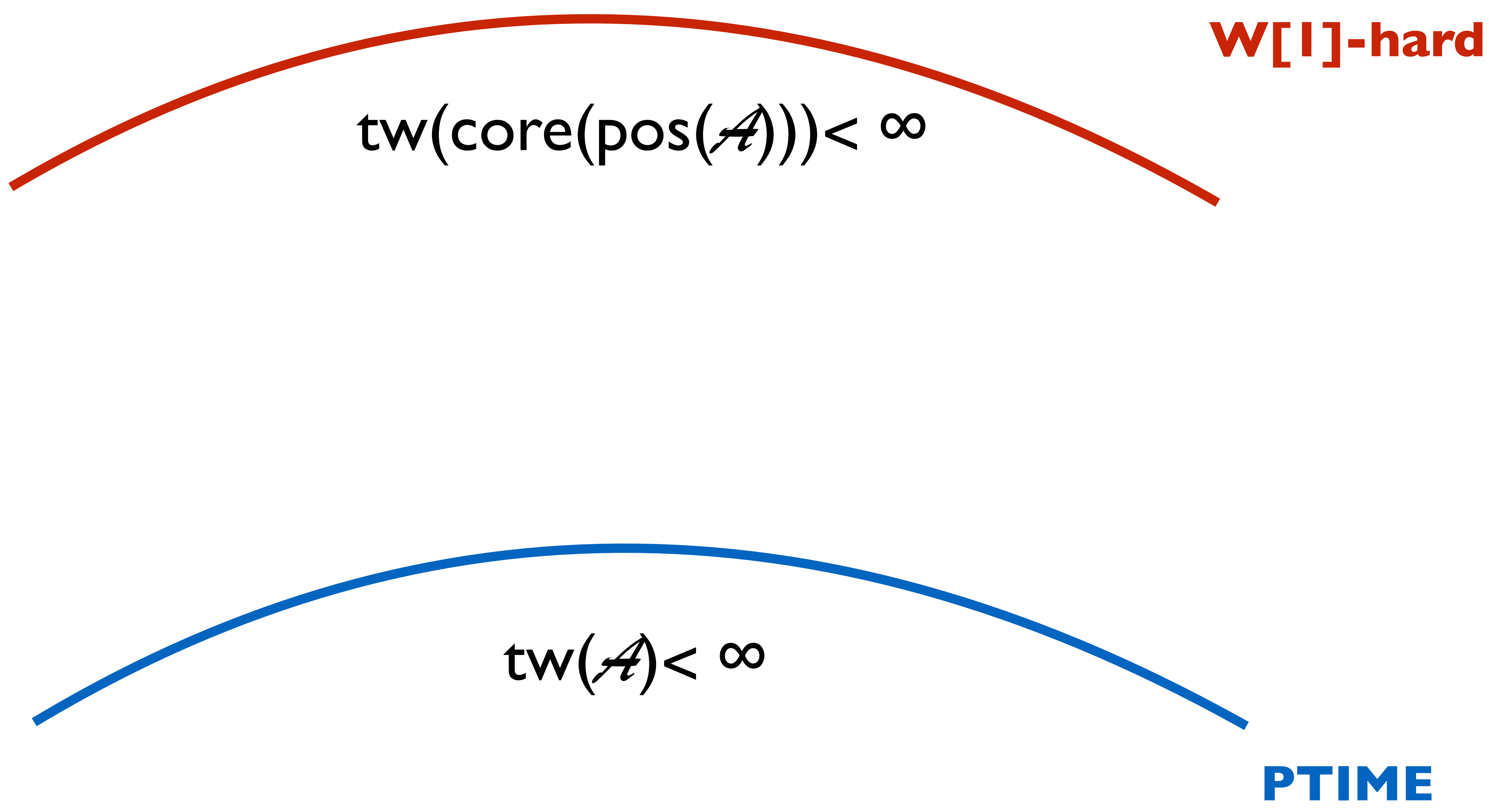
CSP($\mathcal{A}$,-) **not** in PTIME otherwise *[Grohe JACM'07]*

CSP($\mathcal{A}$,-) with ($\mathbb{Q} \cup \{\infty\}$)-valued fns

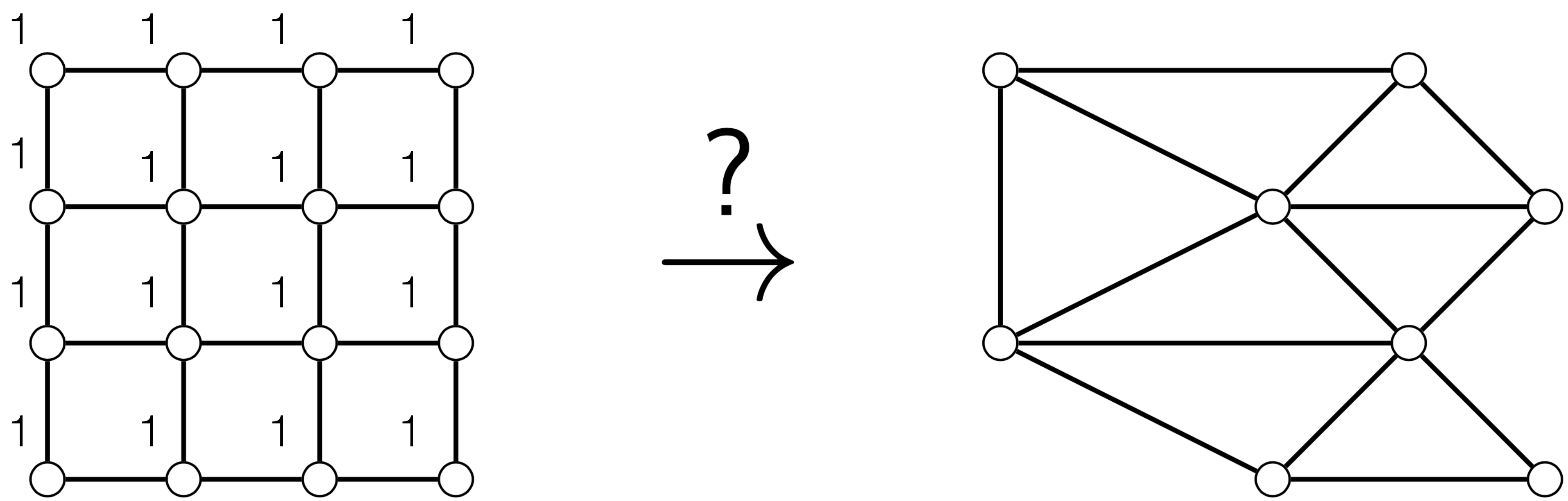
CSP($\mathcal{A}$,-) with ($\mathbb{Q} \cup \{\infty\}$)-valued fns



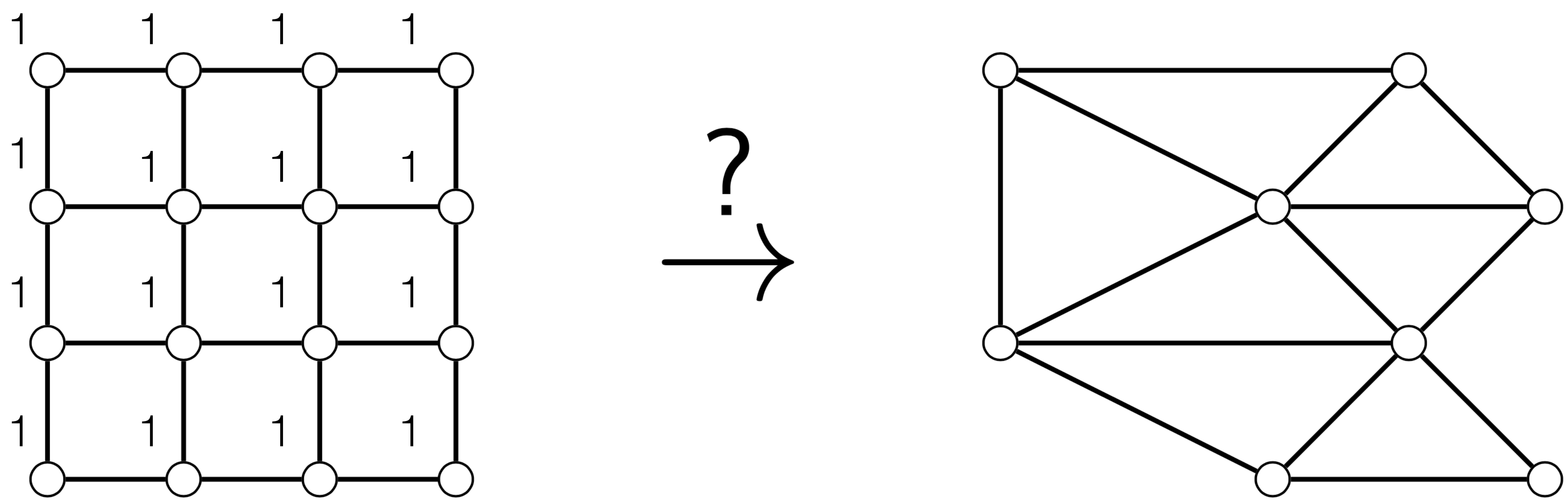
CSP($\mathcal{A}$,-) with ($\mathbb{Q} \cup \{\infty\}$)-valued fns



CSP($\mathcal{A}$,-)

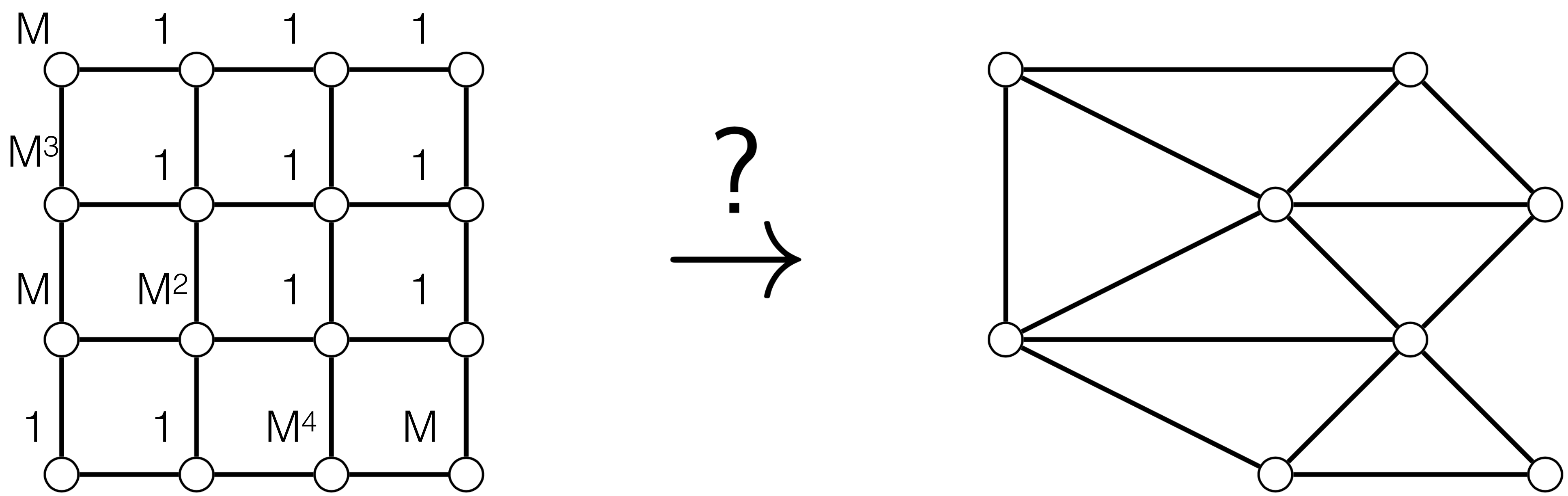


CSP($\mathcal{A}$,-)

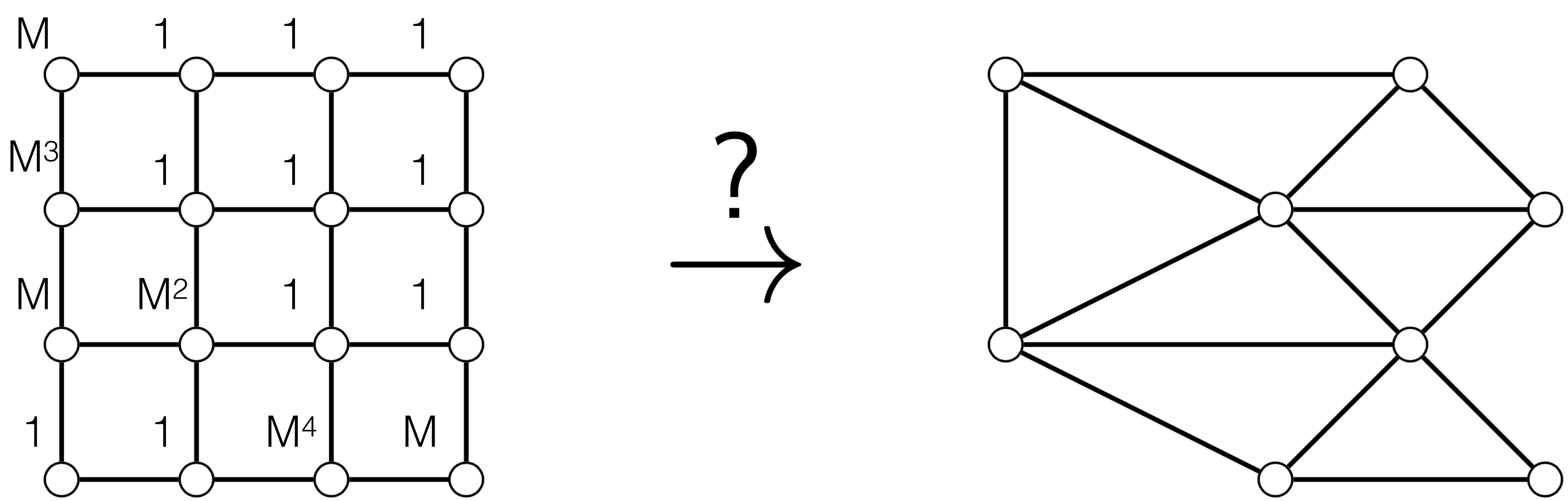


CSP($\mathcal{A}$,-) in PTIME but tw($\mathcal{A}$) = ∞

CSP($\mathcal{A}$,-)



CSP($\mathcal{A}$,-)



CSP($\mathcal{A}$,-) not in PTIME but tw(pos($\mathcal{A}$)) < ∞

CSP($\mathcal{A}$,-) with ($\mathbb{Q}\cup\{\infty\}$)-valued fns

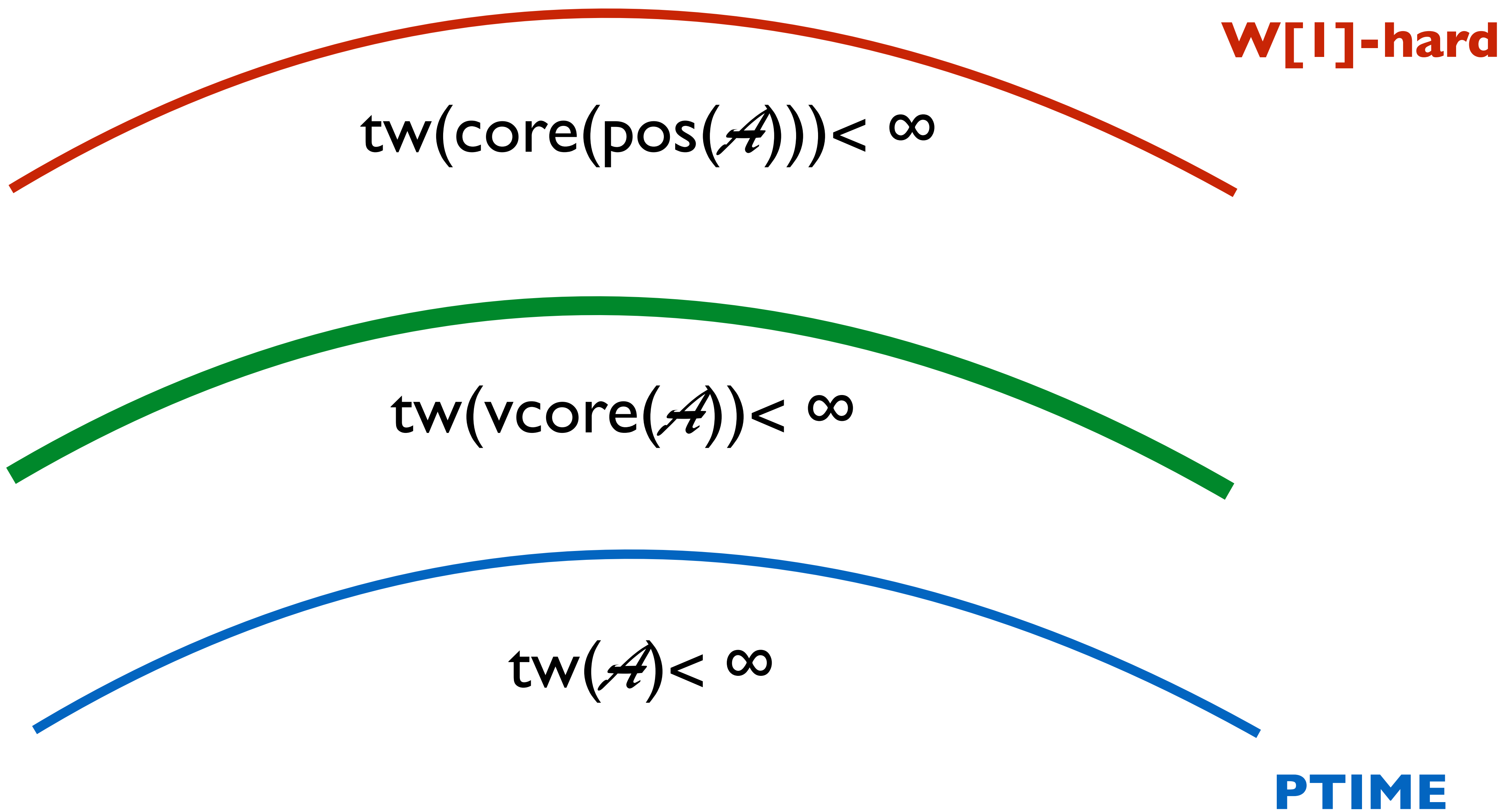
W[1]-hard

tw(core(pos($\mathcal{A}$)))< ∞

tw($\mathcal{A}$)< ∞

PTIME

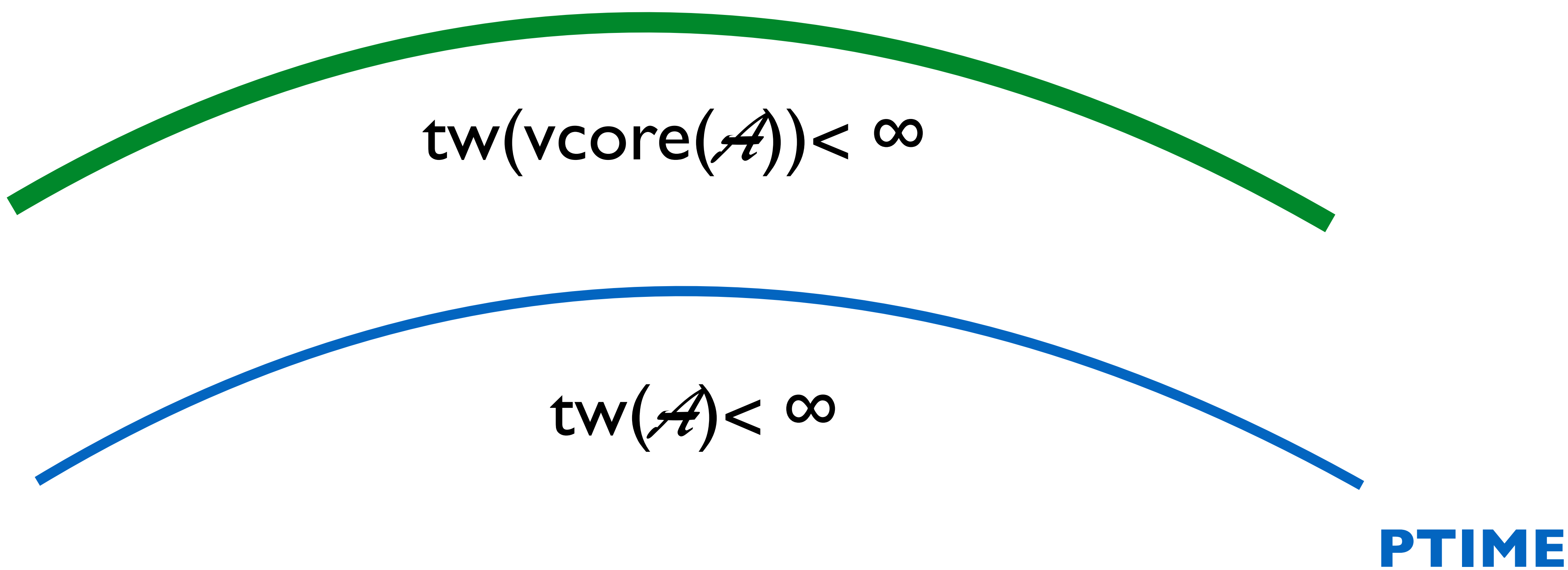
CSP($\mathcal{A}$,-) with ($\mathbb{Q} \cup \{\infty\}$)-valued fns



Theorem [Carbonnel, Romero, Ž. SICOMP'21+]

Assume FPT $\neq$ W[1]. Let $\mathcal{A}$ be a recursively enumerable class of $(\mathbb{Q} \cup \{\infty\})$-valued structures of bounded arity. TFAE:

1. $\mathrm{CSP}(\mathcal{A}, -) \in \mathrm{PTIME}$ $\qquad (|A| + |B|)^{O(1)}$
2. $p\text{-}\mathrm{CSP}(\mathcal{A}, -) \in \mathrm{FPT}$ $\qquad f(|A|) \cdot |B|^{O(1)}$
3. $\mathcal{A}$ is bounded treewidth **modulo valued equivalence**



Sherali-Adams for CSP($\mathscr{A}$,-)

Theorem [Carbonnel, Romero, **Ž**. SICOMP'21+]

Let A be a $(\mathbb{Q} \cup \{\infty\})$-valued σ-structure.
Let $k \geq$ max-arity(σ).
Let A$'$ be a valued core of A.

kth level of Sherali-Adams is exact
$opt_k(\mathrm{A}, \mathrm{B}) = opt(\mathrm{A}, \mathrm{B})$ for every valued σ- structure B

$$\Updownarrow$$

1. $tw(\mathrm{A}') \leq k - 1$

Sherali-Adams for CSP($\mathscr{A}$,-)

Theorem [Carbonnel, Romero, **Ž**. SICOMP'21+]

Let A be a $(\mathbb{Q} \cup \{\infty\})$-valued σ-structure.
Let $k \geq 1$.
Let A′ be a valued core of A.

kth level of Sherali-Adams is exact
$opt_k(\mathrm{A}, \mathrm{B}) = opt(\mathrm{A}, \mathrm{B})$ for every valued σ- structure B

$$\Updownarrow$$

1. $tw_{ms}(\mathrm{A}') \leq k - 1$
2. overlap of A′ is $\leq k$

Exact Solvability

Power of Basic LP *[Kolmogorov, Thapper, Ž. SICOMP'15]*

Power of Sherali-Adams LP *[Thapper, Ž. SICOMP'17]*

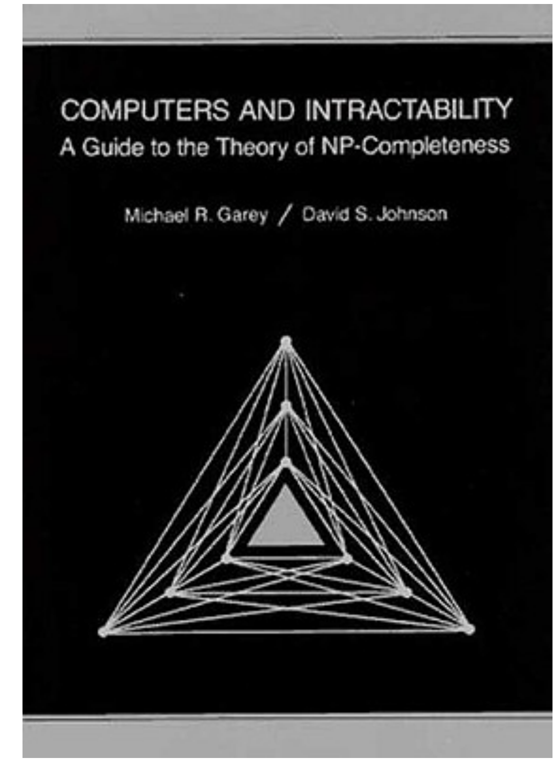
Power of Sum-of-Squares SDP *[Thapper, Ž. ACM ToCT'18]*

Fixed RHS: CSP(-,B)

Fixed LHS: CSP(A,-)

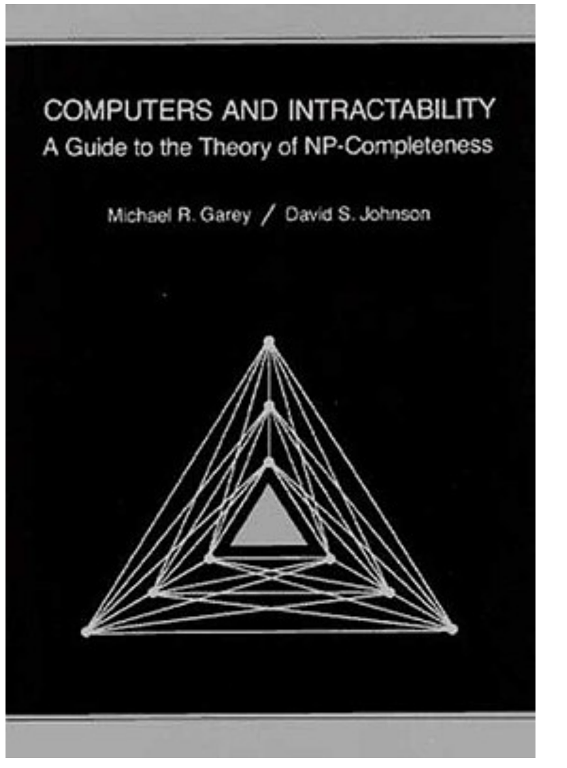
Power of Sherali-Adams LP *[Carbonnel, Migero, Ž. SICOMP'21+]*

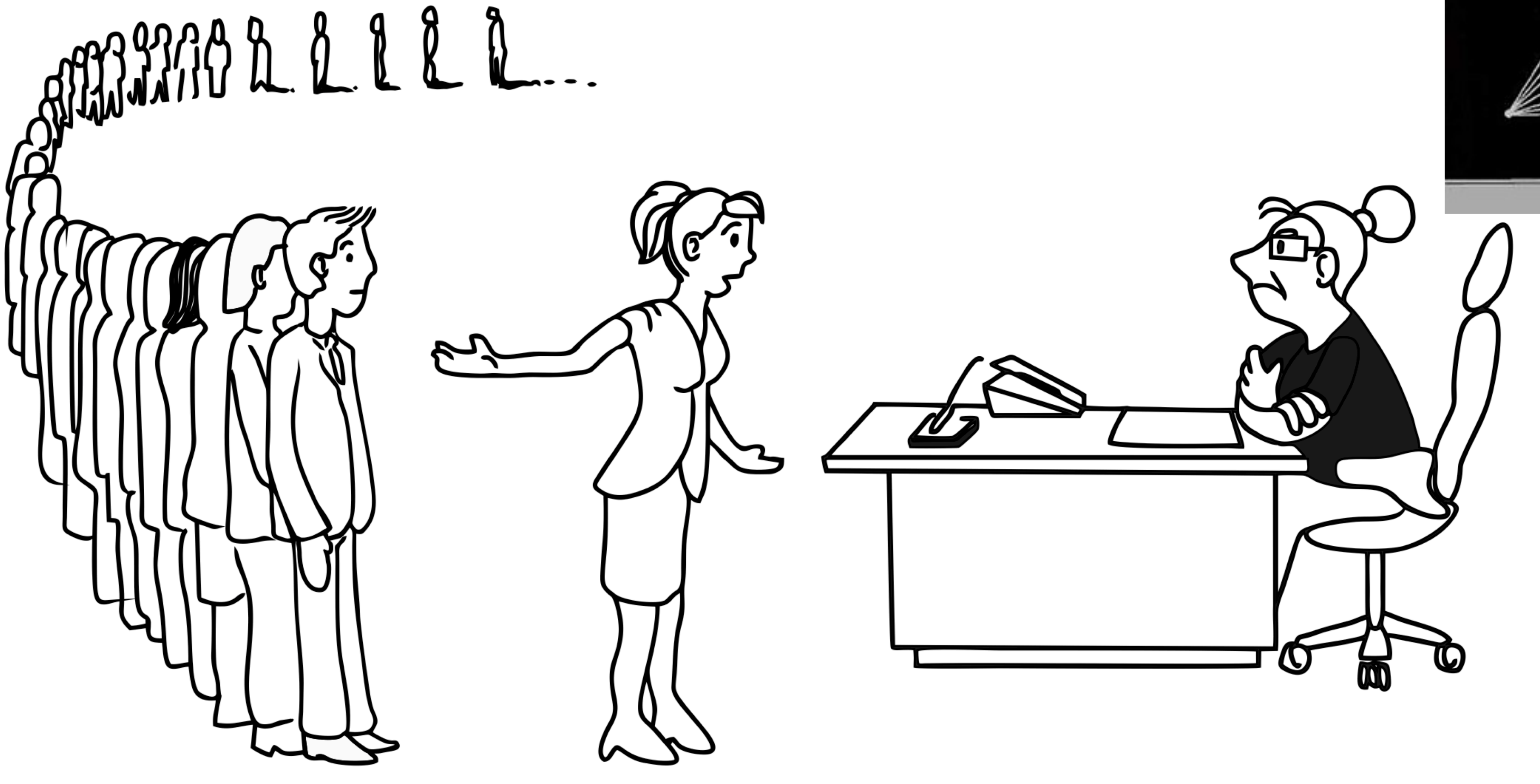
How to Relax



"I can't find an efficient algorithm, but neither can all these famous people."

How to Relax

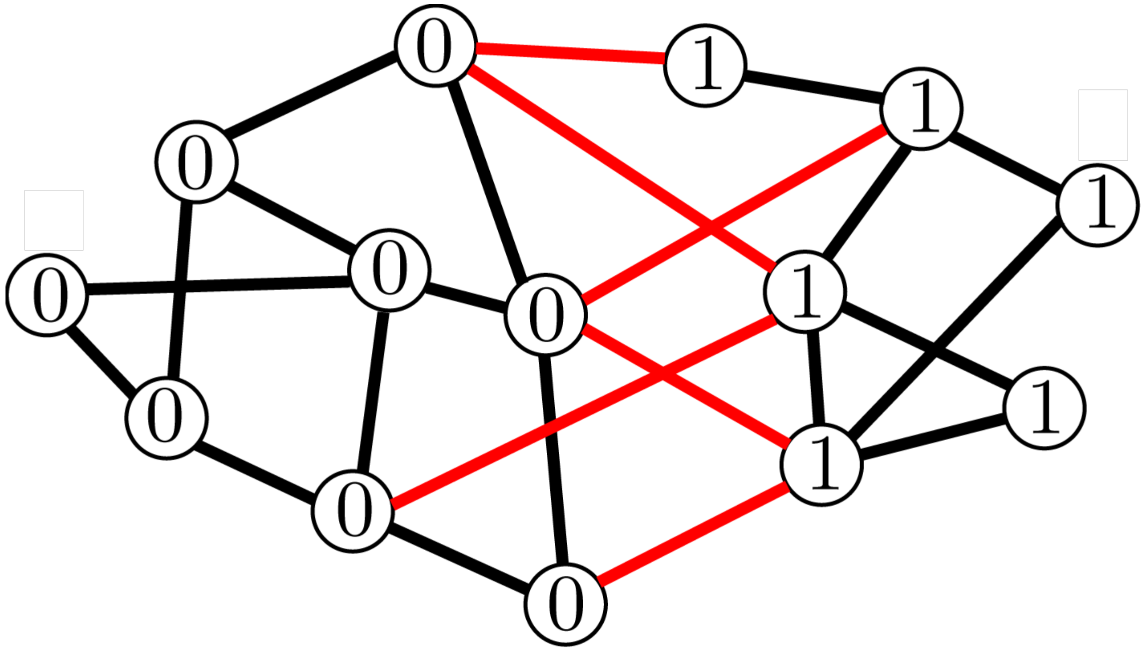




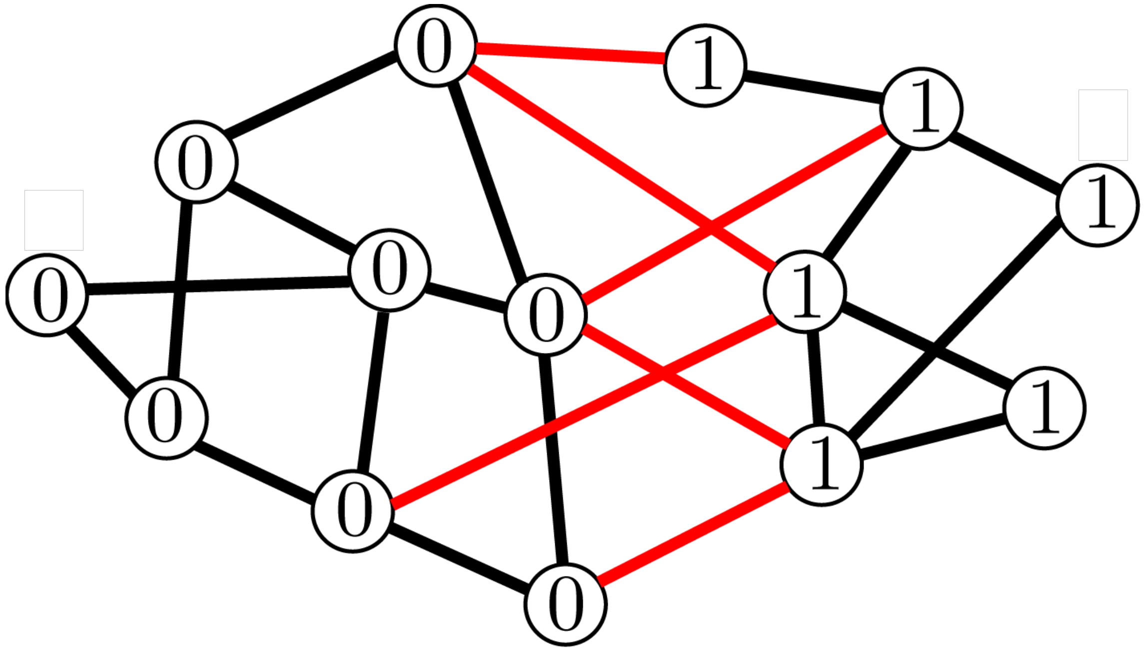
"I can't find an efficient algorithm, but neither can all these famous people."

Satisfy only a fraction of the constraints!

Max-Cut

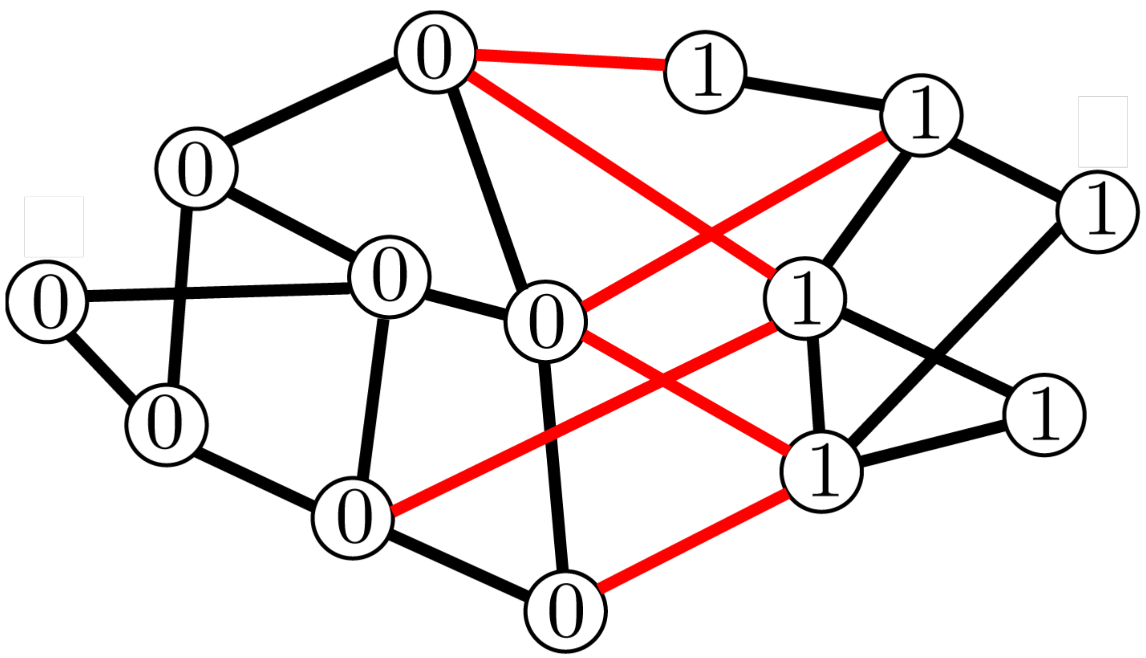


Max-Cut



planar PTIME *[Hadlock SICOMP'75]*

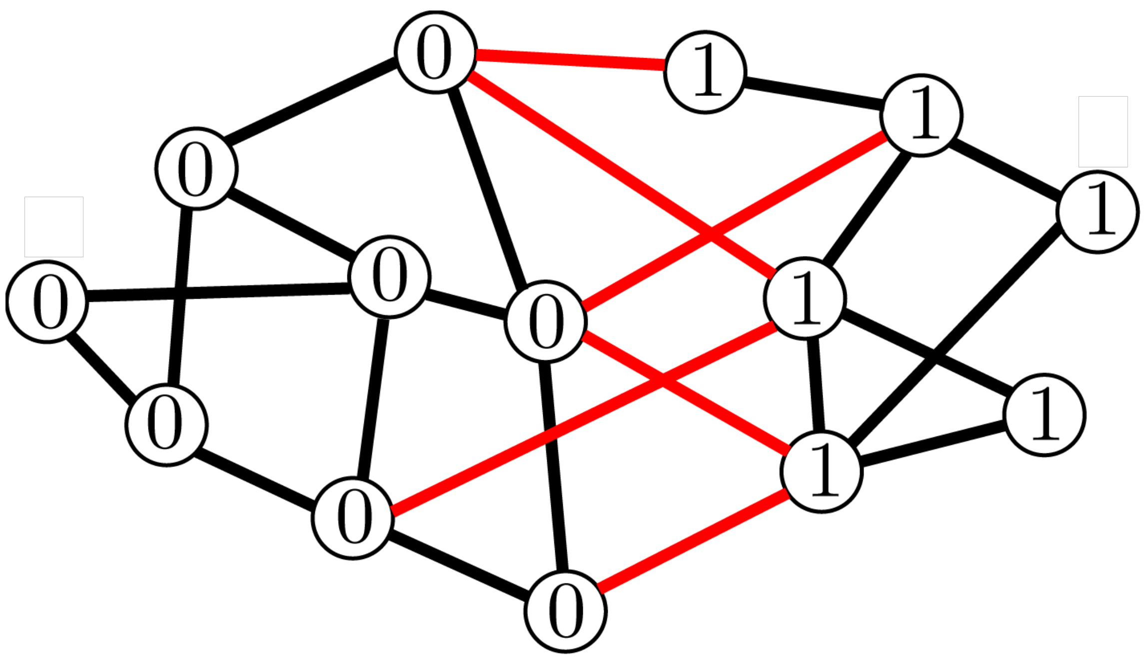
Max-Cut



planar PTIME *[Hadlock SICOMP'75]*

excluded-minor PTAS *[Grohe Combinatorica'03, Demaine et al. FOCS'05]*

Max-Cut

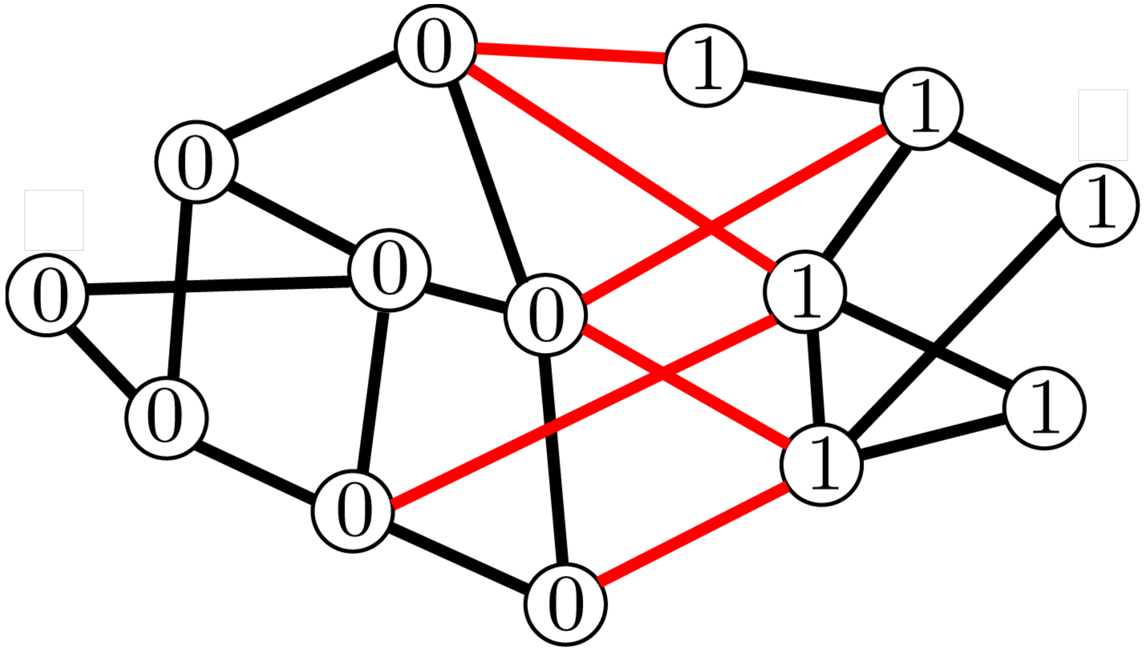


planar PTIME *[Hadlock SICOMP'75]*

excluded-minor PTAS *[Grohe Combinatorica'03, Demaine et al. FOCS'05]*

dense PTAS *[Arora et al. STOC'95, Frieze & Kannan FOCS'96]*

Max-Cut



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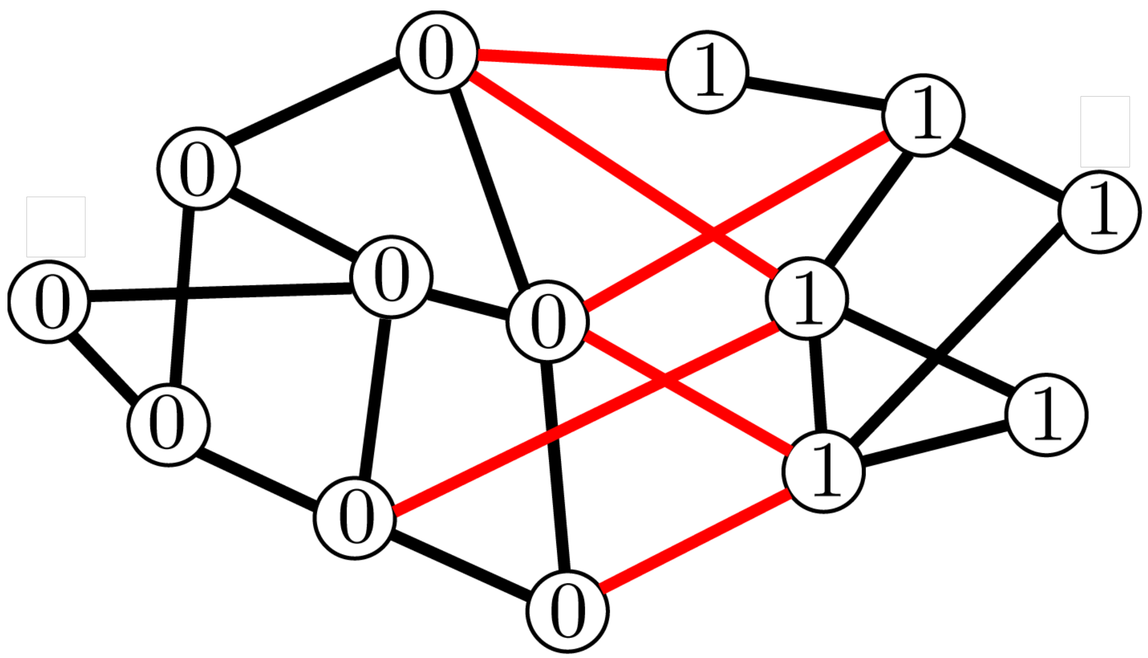
How do you unify sparse and dense graph classes?

Max-Cut



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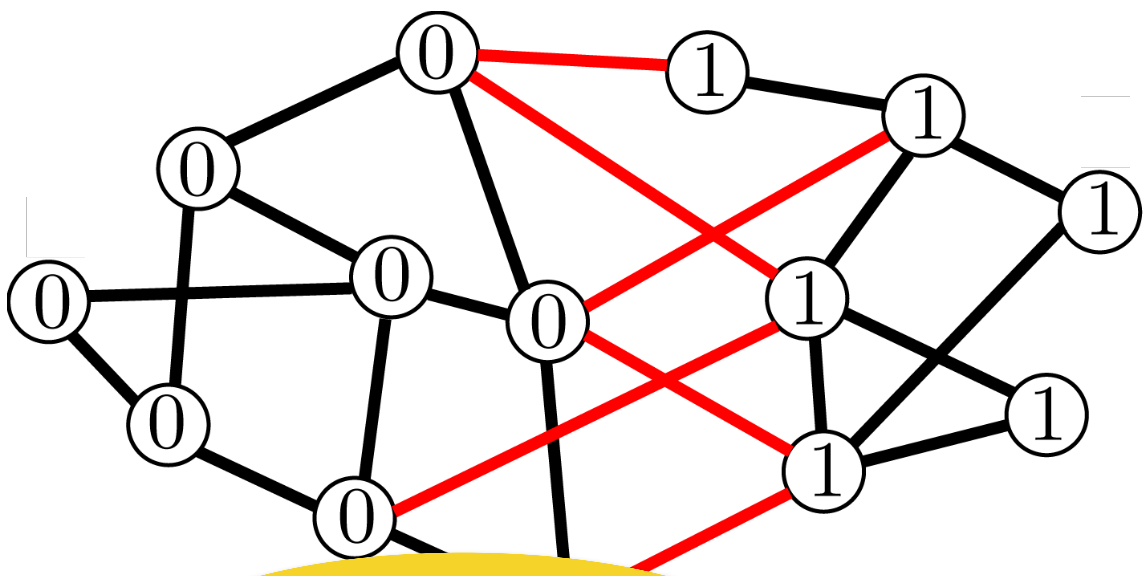
How do you unify sparse and dense graph classes?

Thm: PTAS for Max-CSPs that are treewidth-pliable. *[Romero, Wrochna, Ž. SODA'21]*

Max-Cut



fr-tw-fragility

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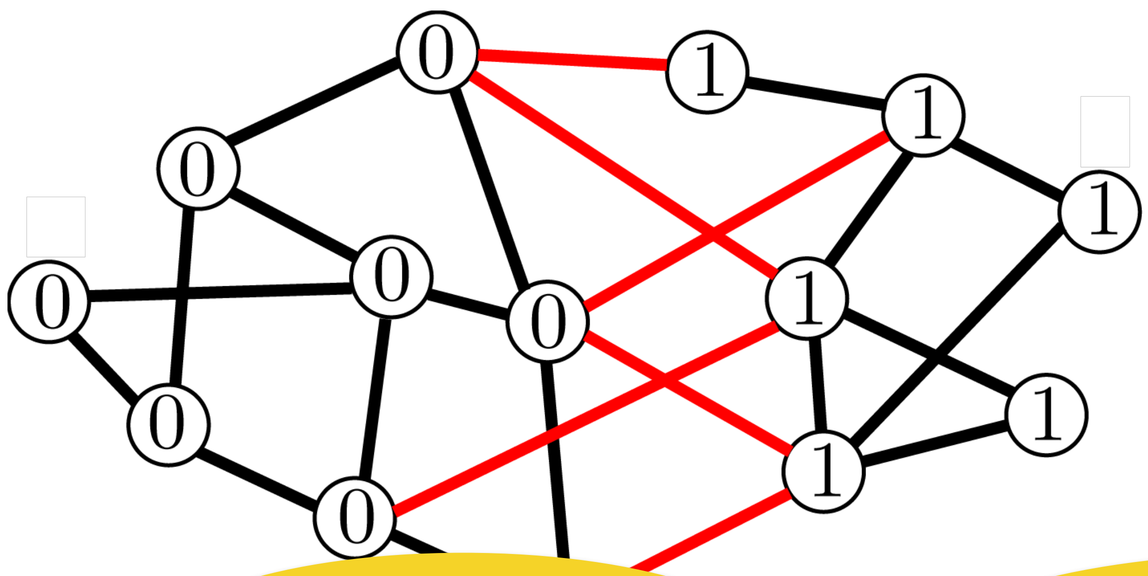

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hyperfiniteness

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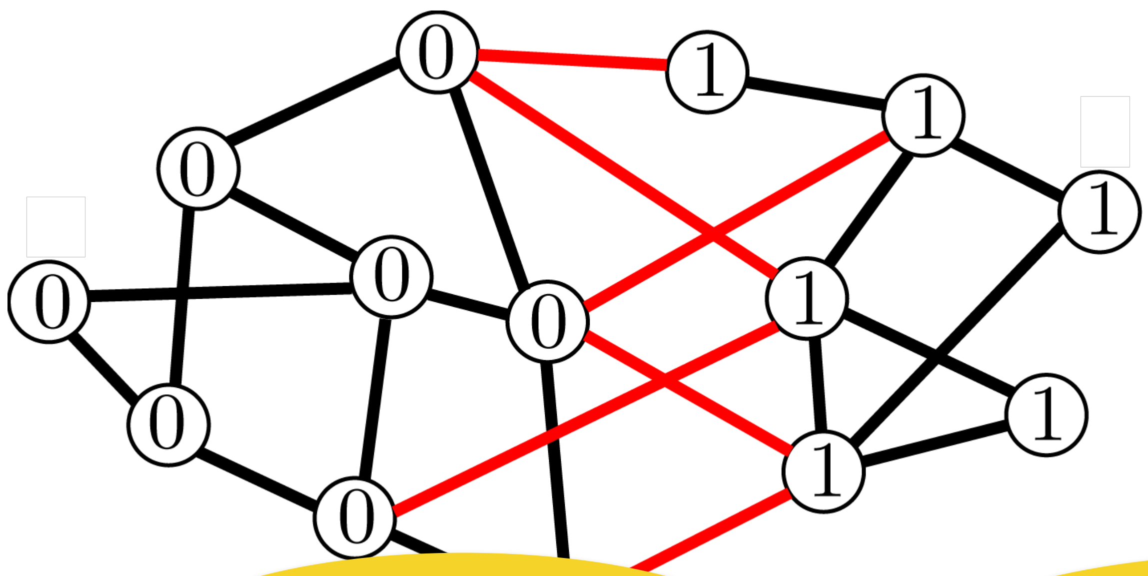

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fr-tw-fragility

hyperfiniteness

regularity partitions

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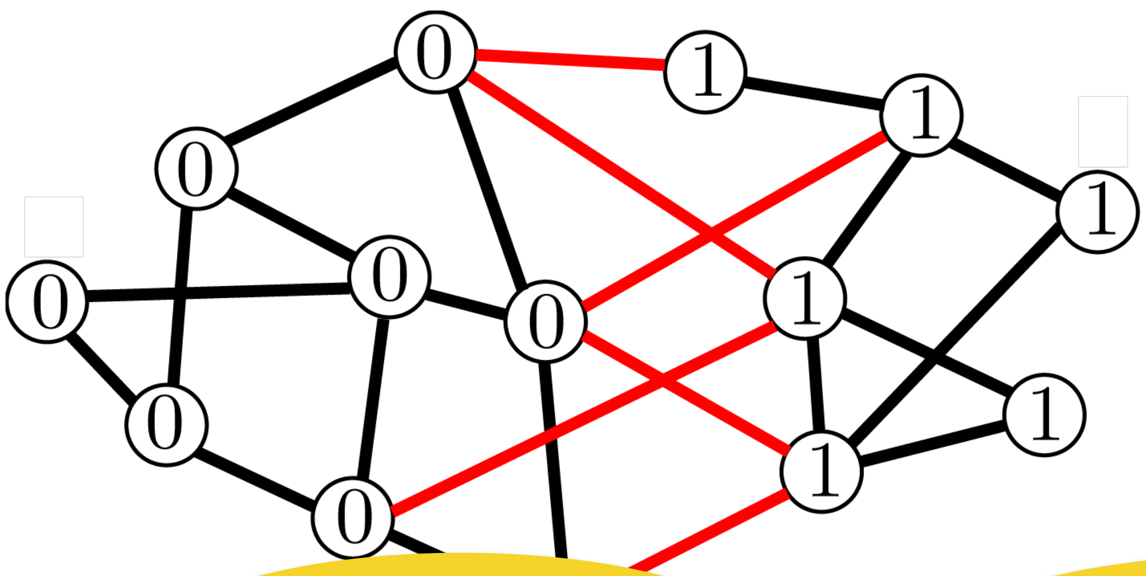

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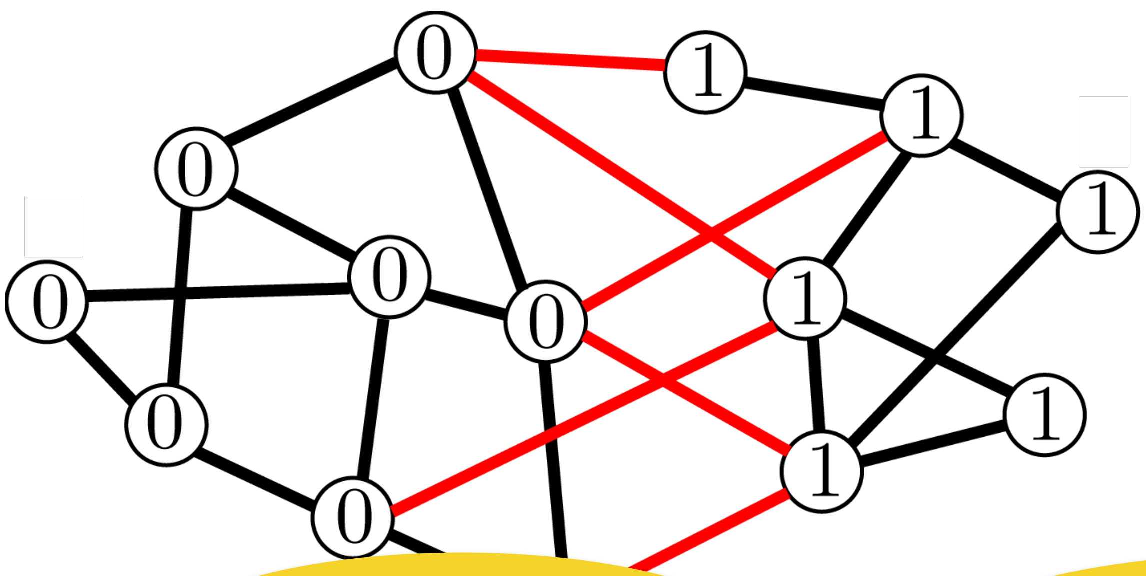
$$\mathscr{C}: \ \forall \varepsilon > 0 \, \exists k \in \mathbb{N} \, \forall G \in \mathscr{C} \, \exists G' \text{ with } \mathrm{tw}(G') \leq k \wedge d_{opt}(G, G') \leq \varepsilon$$

Max-Cut



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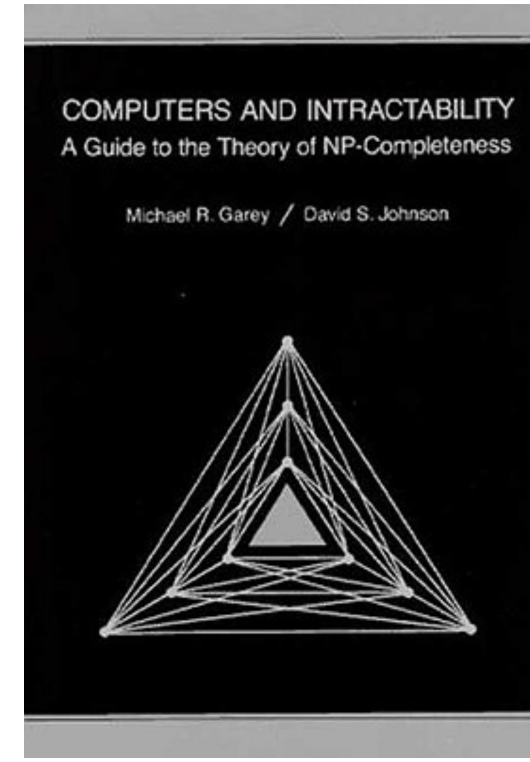
$$\mathscr{C}\colon\ \forall \varepsilon > 0\, \exists k \in \mathbb{N}\, \forall G \in \mathscr{C}\, \exists G' \text{ with } \mathrm{tw}(G') \le k \wedge d_{opt}(G, G') \le \varepsilon$$

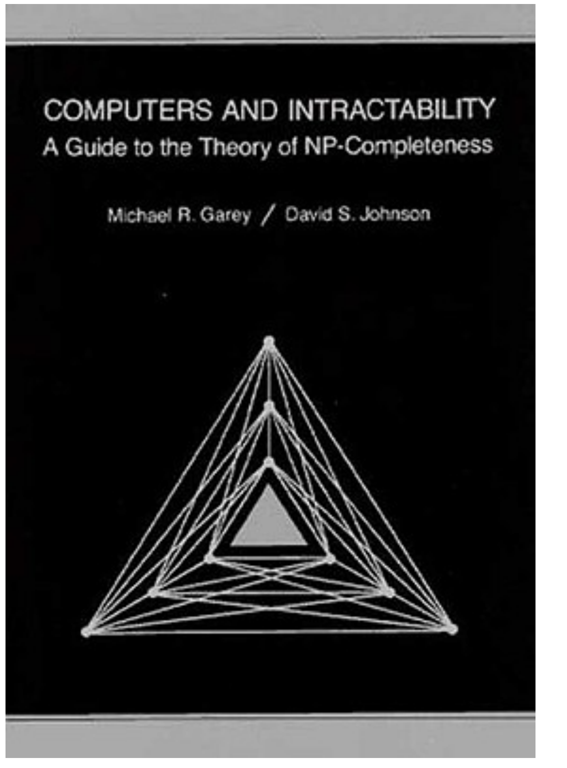
$$d_{opt}(G, G') = \inf_{\varepsilon} \text{Max-Cut}(G) = (1 \pm \varepsilon)\text{Max-Cut}(G')$$

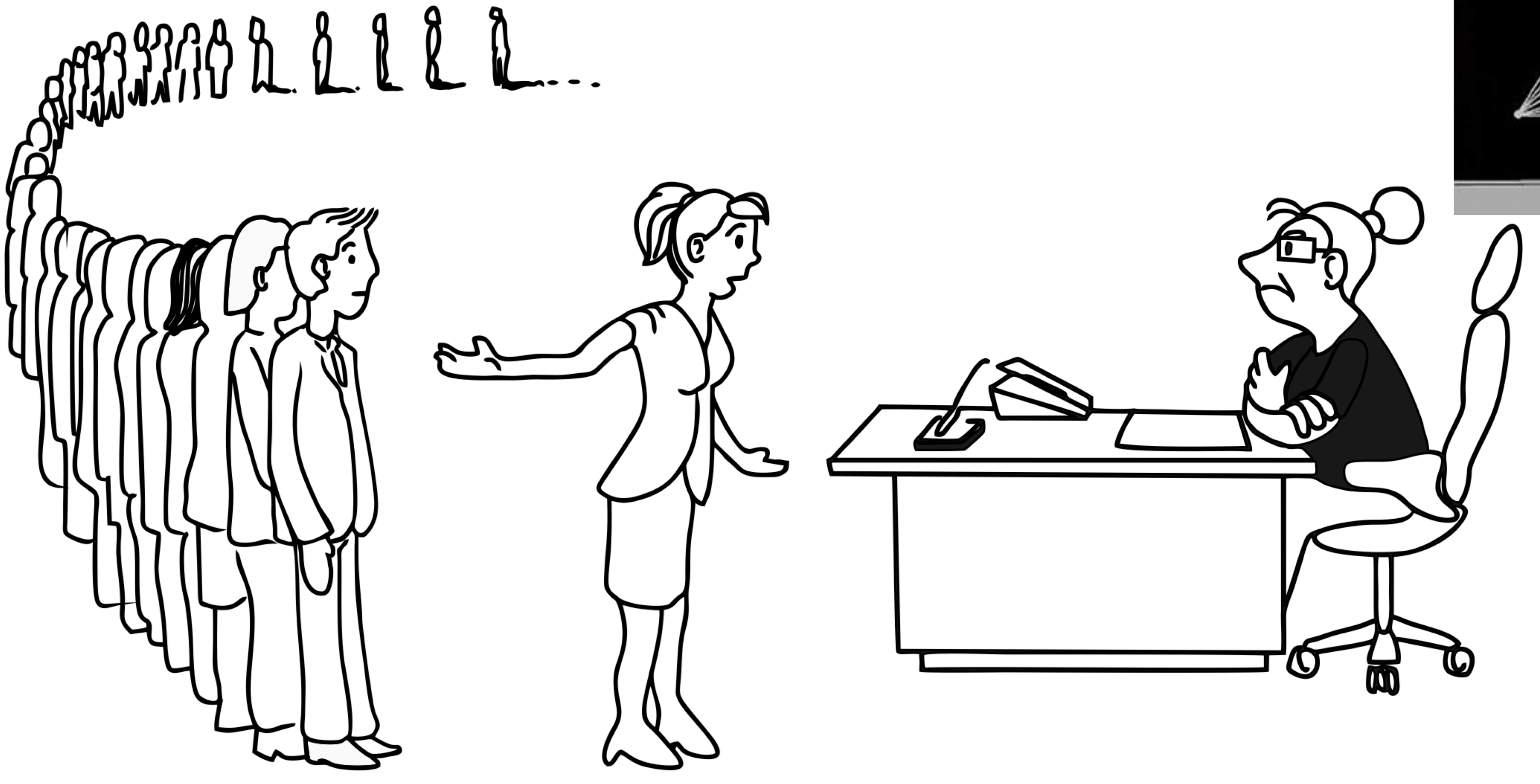
How to Relax



"I can't find an efficient algorithm, but neither can all these famous people."

How to Relax

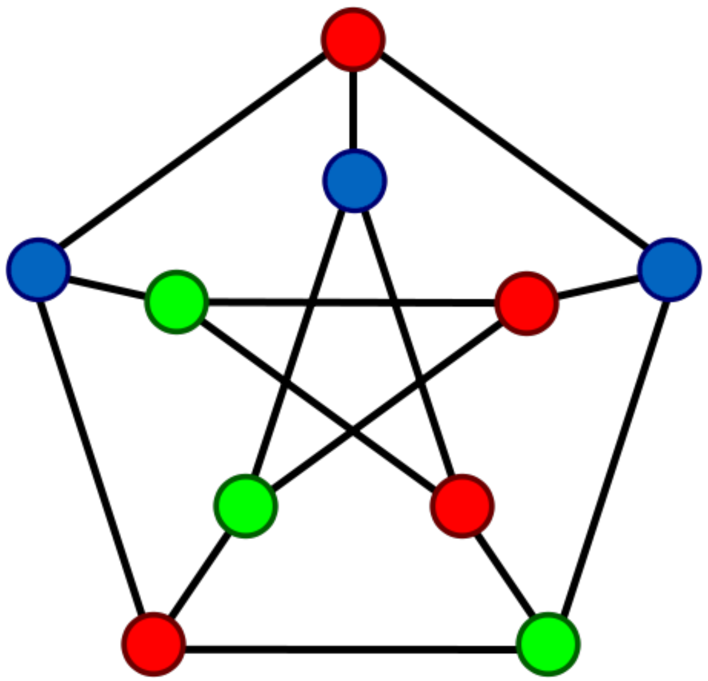




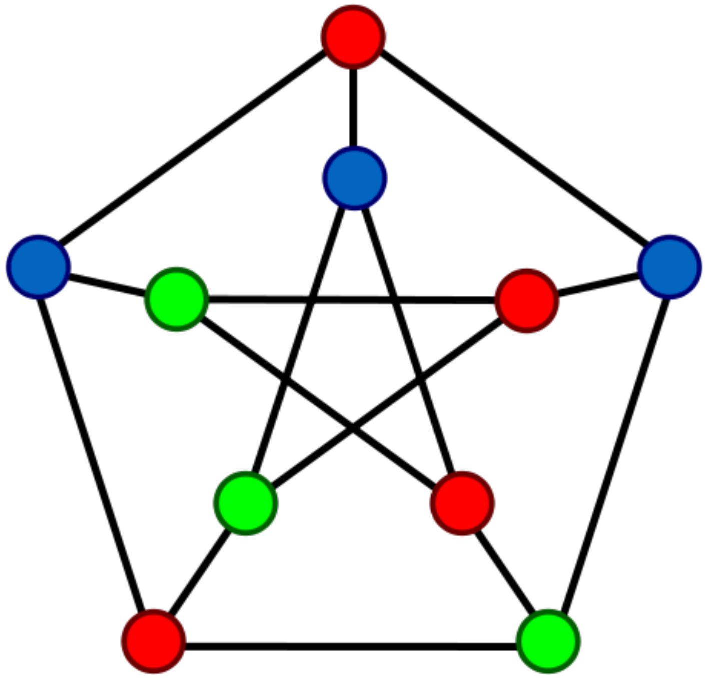
"I can't find an efficient algorithm, but neither can all these famous people."

Satisfy a relaxed version of the constraints!

Approximate Colouring



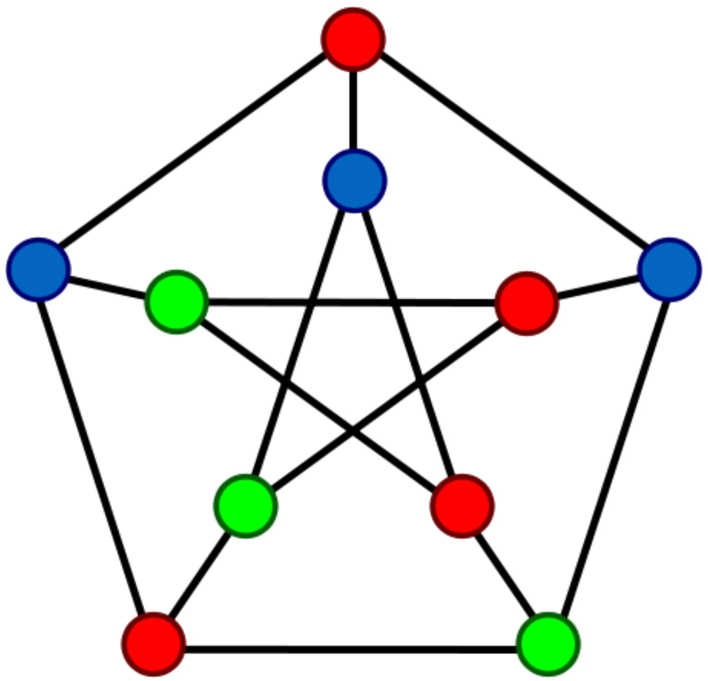
Approximate Colouring



(k,c)-colour

find a c-colouring of a k-colourable graph

Approximate Colouring

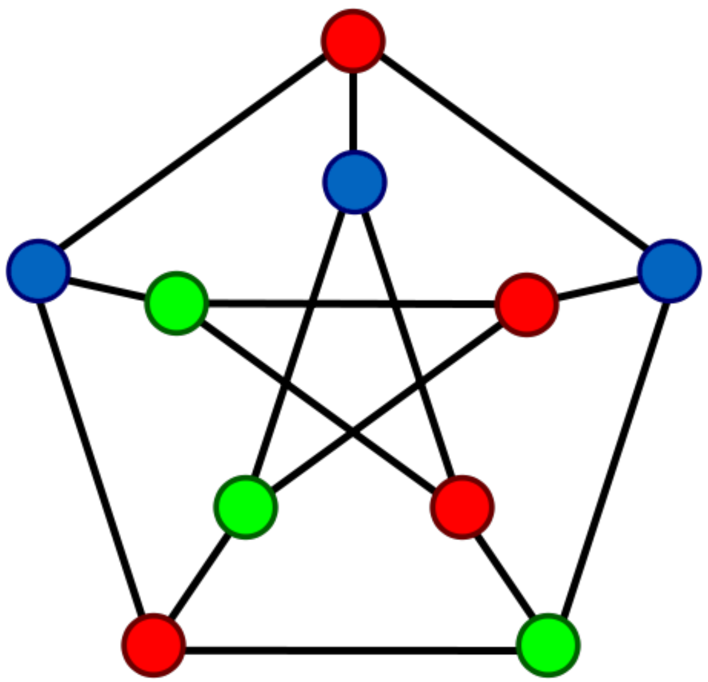


(k,c)-colour

find a c-colouring of a k-colourable graph

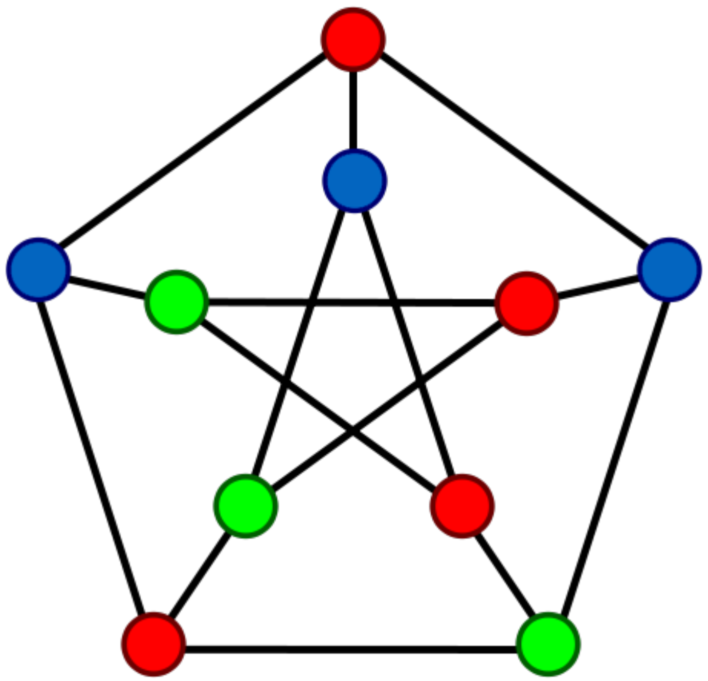
Conjecture: for any constant $3 \leqq k \leqq c$, (k,c)-colour is NP-hard.

Approximate Colouring



Known hardness:

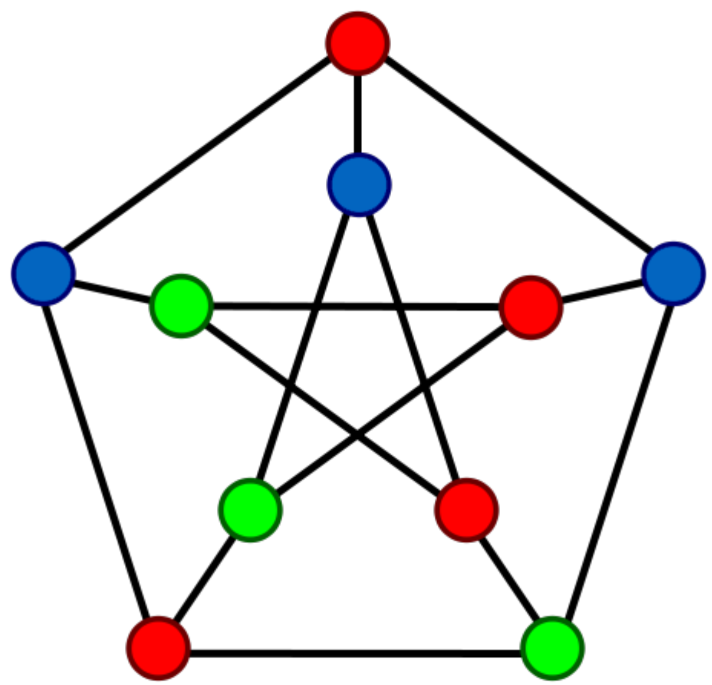
Approximate Colouring



Known hardness:

(k, k) for $k \geq 3$ *[Karp CCC'72]*

Approximate Colouring



Known hardness:

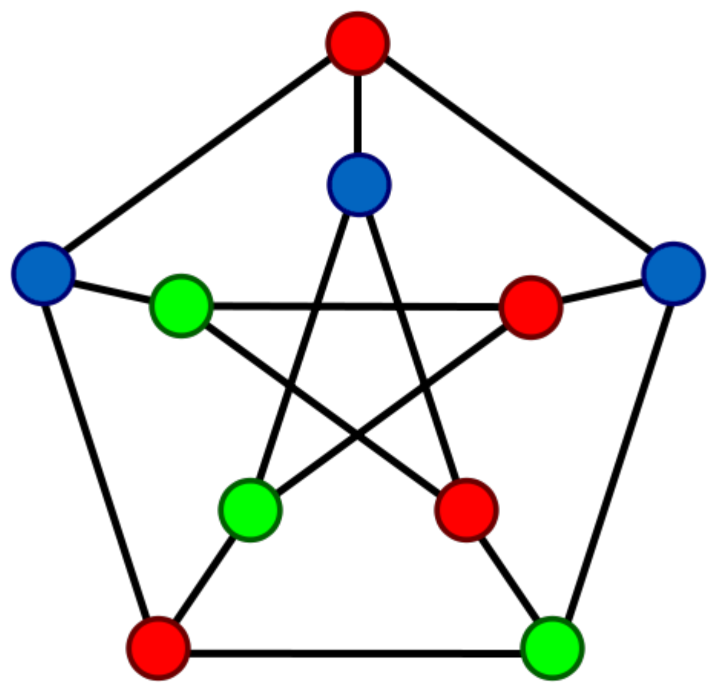
(k, k) for $k \geq 3$ *[Karp CCC'72]*

$(k, 2k-5)$ for $k \geq 6$ *[Garey & Johnson JACM'76]*

$(k, k + 2\lfloor \frac{k}{3} \rfloor - 1)$ for $k \geq 3$ *[Khanna et al. Combinatorica'00]*

$(k, 2k-2)$ for $k \geq 3$ *[Brakensiek & Guruswami CCC'16]*

Approximate Colouring



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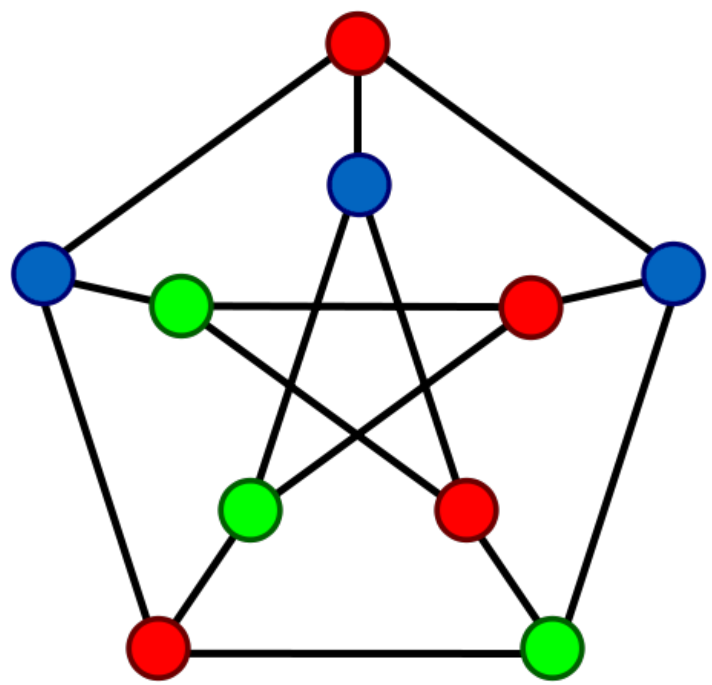
$(k, 2k - 1)$ for $k \geq 3$ *[Bulín et al. STOC'19]*

Approximate Colouring



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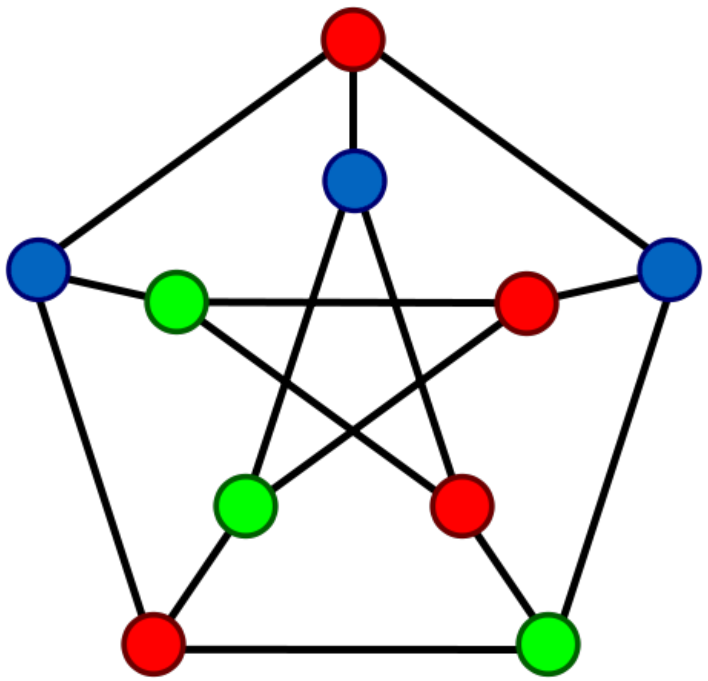
$(k, 2k-1)$ for $k \geq 3$ *[Bulín et al. STOC'19]*

Thm: $(k, \binom{k}{\lfloor k/2 \rfloor} - 1)$ for $k \geq 4.$ *[Wrochna, Ž. SODA'20]*

Approximate Colouring



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Thm: $(k, \binom{k}{\lfloor k/2 \rfloor} - 1)$ for $k \geq 4.$ *[Wrochna, Ž. SODA'20]*

Finding a c-colouring a 3-colourable graph believed to be hard!
(but only known for $3 \leq c \leq 5$)

Approximate Graph Homomorphism

Finding a 3-colouring of a 2-colourable graph easy!

Approximate Graph Homomorphism

Finding a 3-colouring of a 2-colourable graph easy!

Thm: Finding a 3-colouring of $(2+\varepsilon)$-colourable graph is NP-hard.

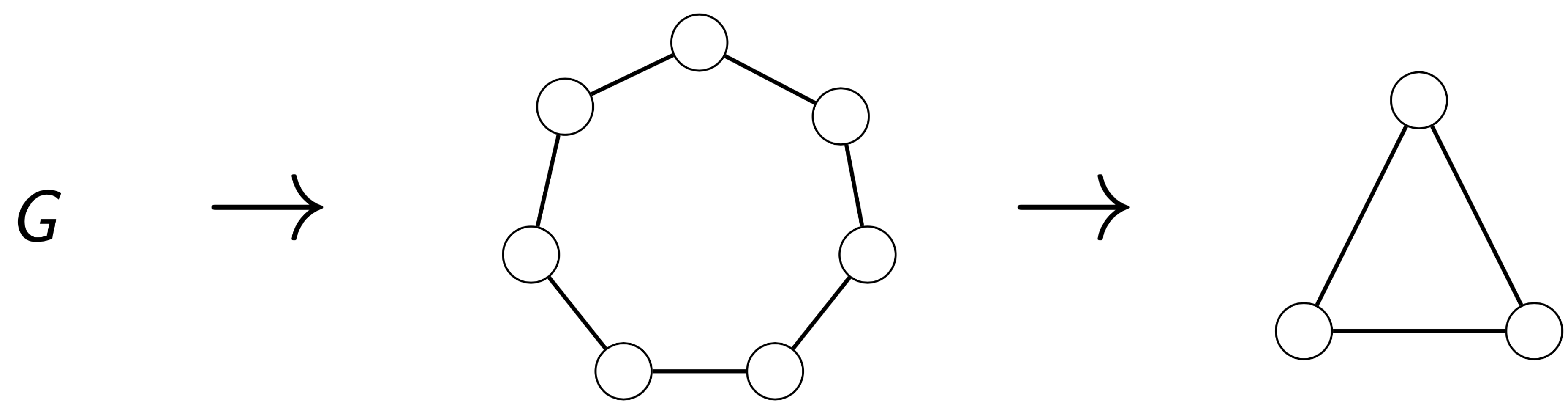
*[Krokhin, Opršal, Wrochna, **Ž**. '21+]*

Approximate Graph Homomorphism

Finding a 3-colouring of a 2-colourable graph easy!

Thm: Finding a 3-colouring of $(2+\varepsilon)$-colourable graph is NP-hard.

*[Krokhin, Opršal, Wrochna, **Ž**. '21+]*



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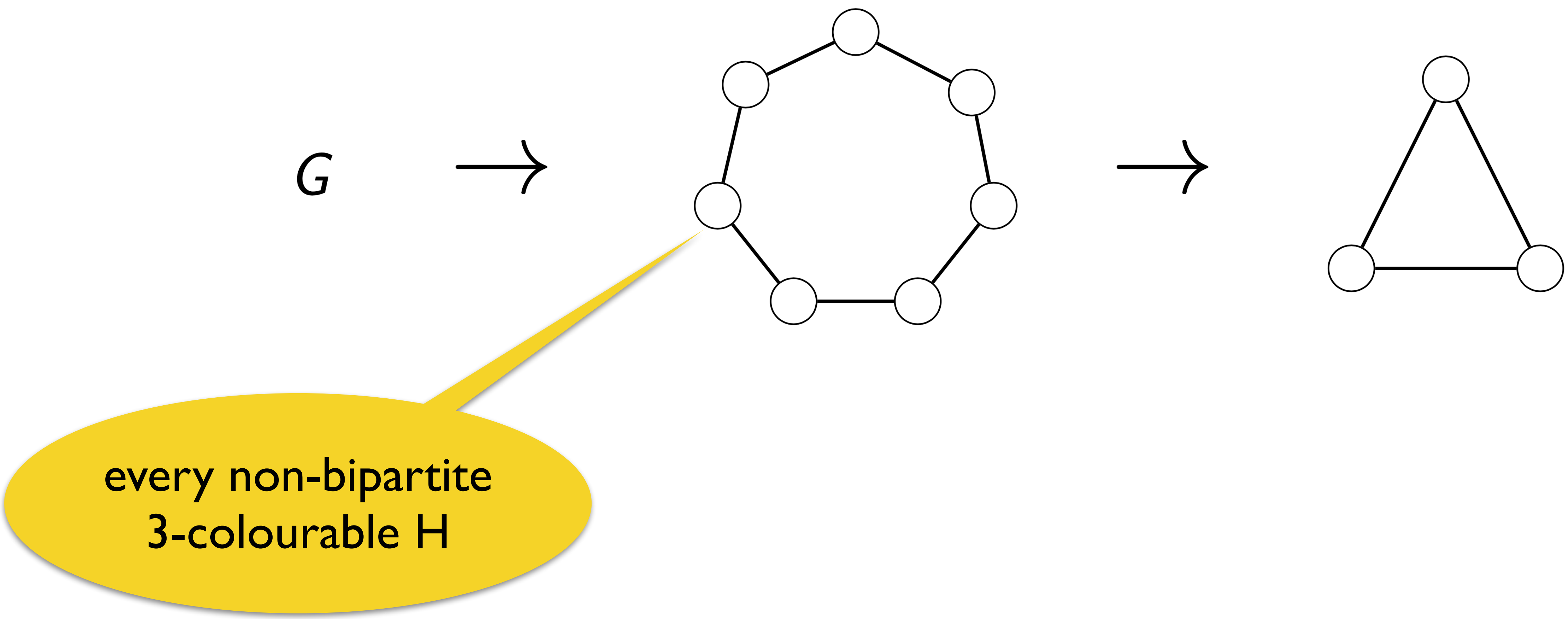

Approximate Graph Homomorphism

4-ε

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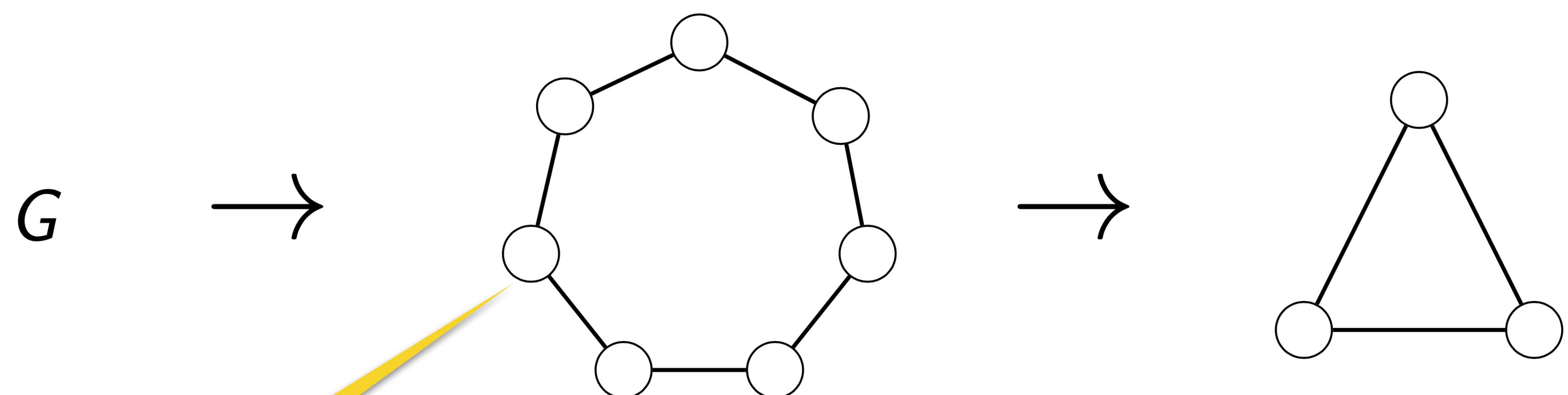

every non-bipartite
3-colourable H

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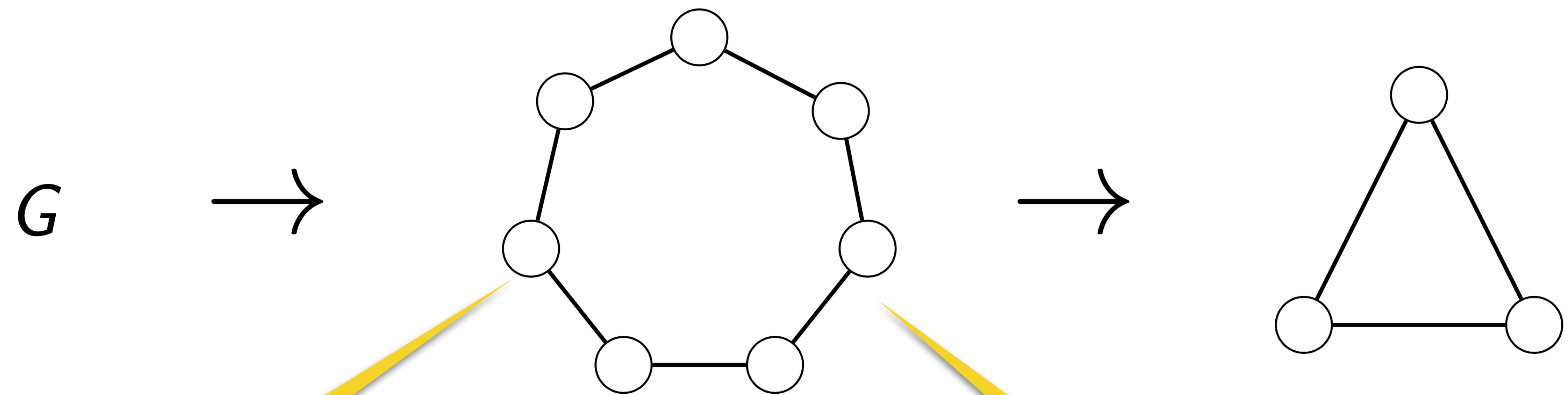

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topology
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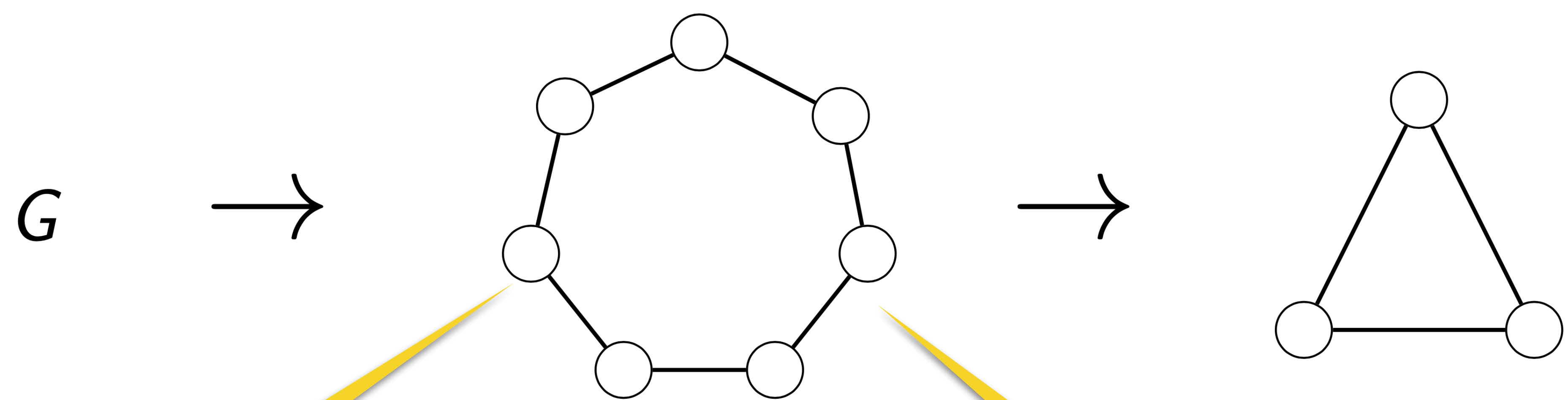

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Thank you!